Nodal finite element de Rham complexes

Kaibo Hu

joint work with Snorre H. Christiansen (Oslo) and Jun Hu (Beijing)

University of Minnesota

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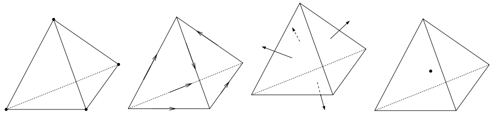
Outline

1. Motivation
2. 2D construction
3. 3D construction

Finite element de Rham complexes



$$\begin{array}{ccccccccccc}
\mathbb{R} & \longrightarrow & H_h(\text{grad}) & \xrightarrow{\text{grad}} & H_h(\text{curl}) & \xrightarrow{\text{curl}} & H_h(\text{div}) & \xrightarrow{\text{div}} & L_h^2 & \longrightarrow & 0 \\
 & & \cap & & \cap & & \cap & & \cap & & \\
\mathbb{R} & \longrightarrow & H(\text{grad}) & \xrightarrow{\text{grad}} & H(\text{curl}) & \xrightarrow{\text{curl}} & H(\text{div}) & \xrightarrow{\text{div}} & L^2 & \longrightarrow & 0
\end{array}$$

where $H(\mathrm{d}) := \left\{u \in L^2 : \mathrm{d}u \in L^2\right\}$.

- applications: diffusion, Maxwell, MHD, Stokes (discontinuous velocity), etc.
- FE: Lagrange, Nédélec (1st & 2nd), Raviart-Thomas, Brezzi-Douglas-Marini etc.
- any dimension, any degree
- software packages: FEniCS, NGSolve etc.
- problems not completely solved (computational aspects):
  - high order bases: condition number, sparsity, symmetry...
  - nodal bases: practitioners' point of view

two relevant issues: good bases are hard to achieve because of the non-scalar nature of the vector elements

Stokes complexes

de Rham complexes with enhanced smoothness (c.f. M. Neilan, talk at IMA 2014)

$$\begin{array}{ccccccccccc}
\mathbb{R} & \xrightarrow{\subset} & H_h^1 & \xrightarrow{\text{grad}} & H_h^+(\text{curl}) & \xrightarrow{\text{curl}} & \left[H_h^1\right]^3 & \xrightarrow{\text{div}} & L_h^2 & \longrightarrow & 0 \\
 & & \cap & & \cap & & \cap & & \cap & & \\
\mathbb{R} & \xrightarrow{\subset} & H^1 & \xrightarrow{\text{grad}} & H^+(\text{curl}) & \xrightarrow{\text{curl}} & \left[H^1\right]^3 & \xrightarrow{\text{div}} & L^2 & \longrightarrow & 0
\end{array}$$

- divergence-free condition is important for fluid computation,

  V.John, A.Linke, C.Merdon, M.Neilan, LG.Rebholz 2017

- Examples of divergence-free Stokes elements:
  - Scott-Vogelius: nodal $\mathscr{P}_r C^0$-$\mathscr{P}_{r-1} C^{-1}$, certain meshes & high degree,

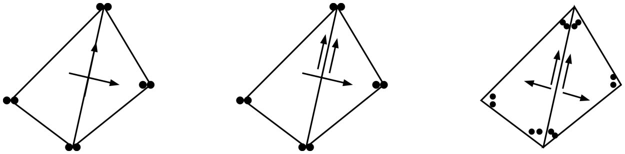
    R. Scott, M. Vogelius 1985; J. Guzmán, R. Scott 2017 etc.
  - Stokes complexes with supersmoothness (super splines),

    R.Falk, M.Neilan 2013; M.Neilan 2015; J.Guzmán, M.Neilan 2018 etc.
  - Stokes pairs (complexes) on macroelements.

    S. Zhang 2008, 2009, 2011; P.Alfeld, T.Sorokina 2015; G.Fu, J.Guzmán, M.Neilan 2018; S.Christiansen, K.Hu 2018 etc.

Goal: $H(\mathrm{curl})/H(\mathrm{div})$ elements with nodal bases fitting into de Rham complexes

2D $H(\mathrm{div})/H(\mathrm{rot})$



- discrete vector fields:
  - continuous element (Scott-Vogelius velocity): continuous normal and tangential components,
  - discontinuous element (DG): discontinuous normal and tangential components,
  - partially discontinuous element: discontinuous tangential, continuous normal.
- various characterizations for the partially discontinuous element:
  - locally $\mathscr{P}_r$, $H(\mathrm{div})$ conforming, C^0 at vertices (super splines),

    R. Stenberg 2010.
  - $\mathscr{P}_r$ Lagrange + BDM_r bubbles,
  - partially discontinuous element and nodal bases.

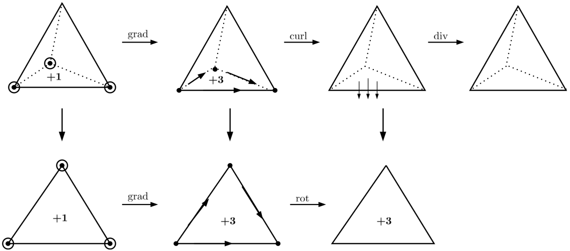
Discrete complexes

- partially discontinuous elements fit in a complex,
- canonical interpolations do not commute with curl/div (or grad/rot),
- bounded cochain projections can be constructed,

  techniques for Stokes elements
- can be *extended* to 3D.

"Restriction" of higher dimensional complex to faces should be a lower dimensional complex.

true for classical de Rham elements, Finite Element System (FES).



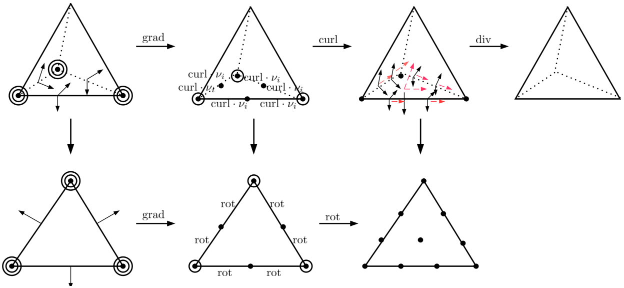
$H(\mathrm{div})$ nodal element

- partially discontinuous Lagrange $\mathscr{P}_r$ vectors, $r \geq 3$,

  Lagrange vectors with discontinuous tangential components on faces

- alternative characterizations
  - continuous normal components across edges and faces, C^0 at vertices,
  - Lagrange $\mathscr{P}_r$ + BDM_r bubbles,

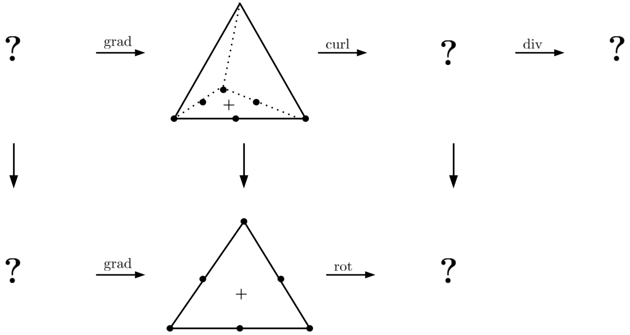
    BDM bubbles have C^0 supersmoothness at vertices



$H(\mathrm{curl})$ nodal elements: motivation

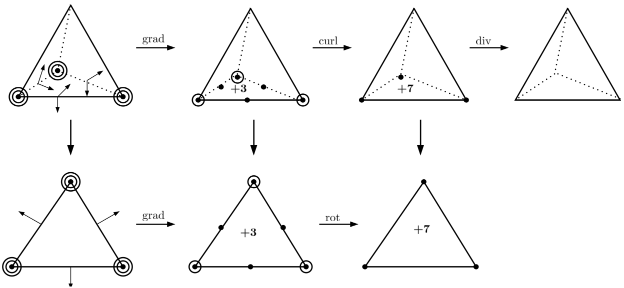
- first attempt: partially discontinuous Lagrange $\mathscr{P}_r$ vectors with possibly discontinuous normals (C^0 on edges, C^τ on faces),
- but... 2D restriction (Scott-Vogelius) unlikely fits into a complex on general meshes.

“Restriction” of a 3D complex to a face should be a reasonable 2D complex



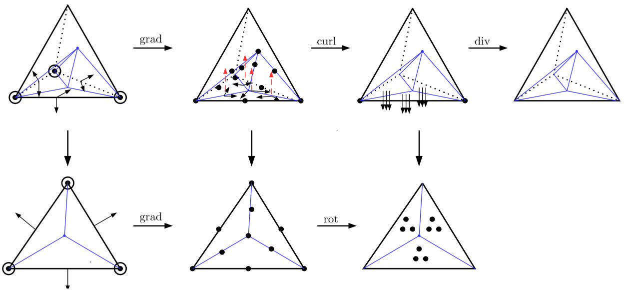
Solution 1: Argyris type complex

- 0-form: C^2 at vertices, C^1 on edges, C^0 across faces,
- 1-form: C^1 at vertices, C^0 on edges, tangential continuity across faces,
  – Hermite type $H(\mathrm{curl})$ nodal element
- 2-form: C^0 at vertices,
- extension of 2D Falk-Neilan Stokes complex.



Solution 2: macroelements on Worsey-Farin split

- 2D restriction: Clough-Tocher complex,
  V.John, A.Linke, C.Merdon, M.Neilan, LG.Rebholz 2017; S. Christiansen, K. Hu, 2018; G. Fu, J. Guzmán, M. Neilan, 2018
- 0-form: C^0 across faces, C^1 elsewhere,
  lowest order with mesh alignment: boils down to the Worsey-Farin C^1 element
- 1-form: tangential continuity across faces, C^0 elsewhere,
  – Lagrange type $H(\mathrm{curl})$ nodal element
- 2-, 3-forms: BDM and DG on the Worsey-Farin split.



Technical issues of macroelements

- classical definition for Worsey-Farin macroelements (based on Bernstein-Bézier techniques):
$$\left\{u \in C^1\left(\mathcal{T}_{\mathrm{WF}}\right) : u \in C^2(w)\right\},$$
$\mathcal{T}_{\mathrm{WF}}$: Worsey-Farin mesh, w: refinement point,
- C^2 condition is *intrinsic supersmoothness*, and can be removed from definition,
  - another example: 2D Clough-Tocher element
  - byproduct of a dimension count,
  - a systematic investigation:

    M. Floater, K. Hu, A characterization of supersmoothness of multivariate splines; in preparation.
- Locally, BDM elements on the Worsey-Farin split have a C^0 pre-image.

  dimension count

Conclusions

- References:
  - S. Christiansen, J. Hu, K. Hu, Nodal finite element de Rham complexes, Num. Math. 2016.
  - J. Hu, K. Hu, Partially discontinuous nodal elements for $H(\mathrm{curl})$ and $H(\mathrm{div})$, in preparation.
- $H(\mathrm{curl})/H(\mathrm{div})$ conforming elements with nodal (Lagrange/Hermite) bases,
- cubical elements yields nonconforming convergence for H^1-L^2 Stokes problems,
  - A. Gillette, K. Hu, S. Zhang, Nonstandard finite element de Rham complexes on cubical meshes, 2018, preprint.
- connecting complexes into the Bernstein-Gelfand-Gelfand diagrams leads to symmetric/trace-free tensor elements,
- potential applications in high order methods and Maxwell/Stokes problems,
- further questions:
  - $H(\mathrm{curl})$ nodal elements on general meshes,

    "generalized Cartesian elements" in early engineering literature
  - bases and preconditioning.

THANK YOU!