Poincaré operators for elasticity

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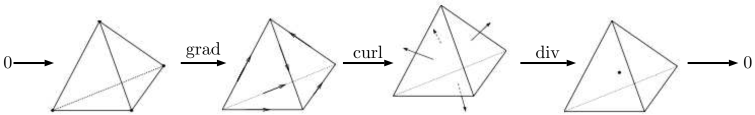
De Rham complex

- de Rham complex

$$0 \longrightarrow C^\infty\Lambda^0 \xrightarrow{d^0} C^\infty\Lambda^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} C^\infty\Lambda^n \longrightarrow 0.$$

  - complex: $d^{i+1}d^i = 0$,
  - local exactness (Poincaré lemma): $d^i v = 0 \Rightarrow v = d^{i-1}\beta$.

- finite element de Rham complex (Whitney, lowest order, 3D)



$$0 \longrightarrow \mathcal{P}_1 \xrightarrow{\text{grad}} \boldsymbol{\mathcal{P}}_0 + \boldsymbol{\mathcal{P}}_0 \times x \xrightarrow{\text{curl}} \boldsymbol{\mathcal{P}}_0 + \mathcal{P}_0 \otimes x \xrightarrow{\text{div}} \mathcal{P}_0 \longrightarrow 0.$$

exact sequence

- how to prove the Poincaré lemma? why these spaces?

Canonical construction of exact sequences

- Poincaré operators (D.G. book; Hiptmair 1999) $\mathfrak{p}^k : C^\infty\Lambda^k \mapsto C^\infty\Lambda^{k-1}$, satisfying
  - null-homotopy property (star-shaped domain):
$$d^{k-1}\mathfrak{p}^k + \mathfrak{p}^{k+1}d^k = \mathrm{id}_{C^\infty\Lambda^k},$$
  - complex property:
$$\mathfrak{p}^{k-1} \circ \mathfrak{p}^k = 0,$$
  - polynomial preserving property:
$$u \in \mathcal{P}_r\Lambda^k \quad \Longrightarrow \quad \mathfrak{p}^k u \in \mathcal{P}_{r+1}\Lambda^{k-1},$$
- Koszul operator (Arnold, Falk, Winther 2006): Poincaré operator acting on homogeneous polynomials, similar properties.
- why leads to exact sequence?
  Algebraic result: if $V^\bullet$ is a complex with both $d^\bullet$ and $\mathfrak{p}^\bullet$, then both $(V^\bullet, d^\bullet)$ and $(V^\bullet, \mathfrak{p}^\bullet)$ are exact.
  $du = 0 \Rightarrow u = d(\mathfrak{p}u)$, $\mathfrak{p}u = 0 \Rightarrow u = \mathfrak{p}(du)$

$$\cdots \rightleftarrows V^{i-1} \underset{\mathfrak{p}^i}{\overset{d^{i-1}}{\rightleftarrows}} V^i \underset{\mathfrak{p}^{i+1}}{\overset{d^i}{\rightleftarrows}} V^{i+1} \rightleftarrows \cdots$$

- construction of $\mathfrak{p}^k$: given a base point W and choose a path $\gamma(t) = W + t(x - W)$,

$$(\mathfrak{p}_W u)_x(\xi_2 \ldots, \xi_k) = \int_0^1 t^{k-1} u_{W+t(x-W)}(x - W, \xi_2, \ldots, \xi_k)\, dt.$$

integration of the contraction operator, relation to Cartan's magic formula

- 3D vector proxy (with $W = 0$):

$$\mathfrak{p}_1 u = \int_0^1 u_{tx} \cdot x dt, \quad \mathfrak{p}_2 v = \int_0^1 t v_{tx} \wedge x dt, \quad \mathfrak{p}_3 w = \int_0^1 t^2 w_{tx} x dt.$$

Example of polynomial de Rham complexes

- construction of $\mathcal{P}_r^-\Lambda^k$ complexes:
  - input: complexes with $d^\bullet$ but not with $\mathfrak{p}^\bullet$

$$\cdots \longrightarrow \mathcal{P}_r\Lambda^{i-1} \xrightarrow{d^{i-1}} \mathcal{P}_r\Lambda^i \xrightarrow{d^i} \mathcal{P}_r\Lambda^{i+1} \longrightarrow \cdots$$

  - output: exact sequence:

$$\cdots \longrightarrow \mathcal{P}_r\Lambda^{i-1} + \mathfrak{p}\mathcal{P}_r\Lambda^i \xrightarrow{d^{i-1}} \mathcal{P}_r\Lambda^i + \mathfrak{p}\mathcal{P}_r\Lambda^{i+1} \xrightarrow{d^i} \mathcal{P}_r\Lambda^{i+1} + \mathfrak{p}\mathcal{P}_r\Lambda^{i+2} \longrightarrow \cdots$$

- construction of $\mathcal{P}_r\Lambda^k$ complexes:
  - input: complex already closed with both $d^\bullet$ and $\mathfrak{p}^\bullet$

$$\cdots \longrightarrow \mathcal{P}_r\Lambda^{i-1} \xrightarrow{d^{i-1}} \mathcal{P}_{r-1}\Lambda^i \xrightarrow{d^i} \mathcal{P}_{r-2}\Lambda^{i+1} \longrightarrow \cdots$$

$$\cdots \longleftarrow \mathcal{P}_r\Lambda^{i-1} \xleftarrow{\mathfrak{p}^i} \mathcal{P}_{r-1}\Lambda^i \xleftarrow{\mathfrak{p}^{i+1}} \mathcal{P}_{r-2}\Lambda^{i+1} \longleftarrow \cdots$$

  - output: invariant

Question

displacement formulation

Krôner's continuum description of dislocations/defects, internal stress

$$0 \longrightarrow C^\infty \otimes \mathbb{V} \xrightarrow{\ \mathrm{def}\ } C^\infty \otimes \mathbb{S} \xrightarrow{\ \mathrm{inc}\ } C^\infty \otimes \mathbb{S} \xrightarrow{\ \mathrm{div}\ } C^\infty \otimes \mathbb{V} \longrightarrow 0$$

$$\mathit{displacement} \qquad \mathit{strain}\ (\mathit{metric}) \qquad \mathit{stress}\ (\mathit{curvature}) \qquad \mathit{force}$$

intrinsic elasticity (Ciarlet et al.)

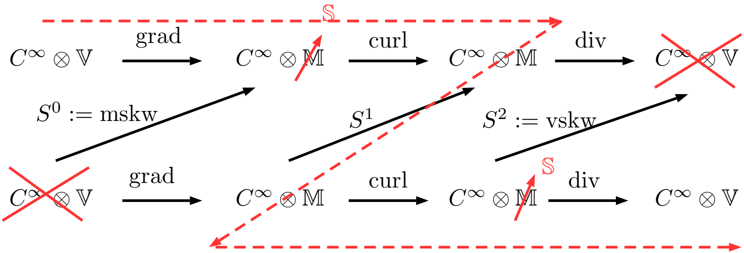
Hellinger-Reissner principle of elasticity

$\mathbb{V}$: vectors $\quad$ $\mathbb{S}$: symmetric matrices $\quad$ $\mathrm{def} = \mathrm{sym\,grad}$, $\quad$ $\mathrm{inc} := \mathrm{curl} \circ \mathrm{T} \circ \mathrm{curl}$

Question: Poincaré type operators $\mathscr{P}^\bullet$ for the elasticity complex?

- homotopy identity $\mathscr{D}\mathscr{P} + \mathscr{P}\mathscr{D} = \mathrm{id}$, ($\mathscr{D}^\bullet = \mathrm{def}, \mathrm{inc}, \mathrm{div}$),
- complex property $\mathscr{P}^2 = 0$,
- polynomial preserving property.

Bernstein-Gelfand-Gelfand type construction: intuitive ideas



$$0 \longrightarrow C^\infty \otimes \mathbb{V} \xrightarrow{\text{def}} C^\infty \otimes \mathbb{S} \xrightarrow{\text{curl}}$$

$$\xrightarrow{\text{curl}^T} C^\infty \otimes \mathbb{S} \xrightarrow{\text{div}} C^\infty \otimes \mathbb{V} \longrightarrow 0.$$

Explicit way of doing homological algebra is needed.

Explicit projections

$$
\begin{tikzcd}[column sep=huge, row sep=large]
\begin{pmatrix} C^\infty \otimes \mathbb{V} \\ C^\infty \otimes \mathbb{V} \end{pmatrix} \arrow[r, "\begin{pmatrix} d & \\ & d \end{pmatrix}"] \arrow[d, "\begin{pmatrix} I & K \\ & I \end{pmatrix}"'] & \begin{pmatrix} C^\infty \otimes \mathbb{M} \\ C^\infty \otimes \mathbb{M} \end{pmatrix} \arrow[r, "\begin{pmatrix} d & \\ & d \end{pmatrix}"] \arrow[d] & \begin{pmatrix} C^\infty \otimes \mathbb{M} \\ C^\infty \otimes \mathbb{M} \end{pmatrix} \arrow[r, "\begin{pmatrix} d & \\ & d \end{pmatrix}"] \arrow[d] & \begin{pmatrix} C^\infty \otimes \mathbb{V} \\ C^\infty \otimes \mathbb{V} \end{pmatrix} \arrow[d] \\
\begin{pmatrix} C^\infty \otimes \mathbb{V} \\ C^\infty \otimes \mathbb{V} \end{pmatrix} \arrow[r, "\begin{pmatrix} d & -S \\ 0 & d \end{pmatrix}"] \arrow[d, "\Pi^0"'] & \begin{pmatrix} C^\infty \otimes \mathbb{M} \\ C^\infty \otimes \mathbb{M} \end{pmatrix} \arrow[r, "\begin{pmatrix} d & -S \\ 0 & d \end{pmatrix}"] \arrow[d, "\Pi^1"'] & \begin{pmatrix} C^\infty \otimes \mathbb{M} \\ C^\infty \otimes \mathbb{M} \end{pmatrix} \arrow[r, "\begin{pmatrix} d & -S \\ 0 & d \end{pmatrix}"] \arrow[d, "\Pi^2"'] & \begin{pmatrix} C^\infty \otimes \mathbb{V} \\ C^\infty \otimes \mathbb{V} \end{pmatrix} \arrow[d, "\Pi^3"'] \\
\Gamma^0 \arrow[r, "\begin{pmatrix} d & -S \\ 0 & d \end{pmatrix}"] \arrow[d, "\mathcal{I}"'] & \Gamma^1 \arrow[r, "\begin{pmatrix} d & -S \\ 0 & d \end{pmatrix}"] \arrow[d, "\mathcal{I}"'] & \Gamma^2 \arrow[r, "\begin{pmatrix} d & -S \\ 0 & d \end{pmatrix}"] \arrow[d, "\mathcal{I}"'] & \Gamma^3 \arrow[d, "\mathcal{I}"'] \\
C^\infty \otimes \mathbb{V} \arrow[r, "\mathrm{def}"] & C^\infty \otimes \mathbb{S} \arrow[r, "\mathrm{inc}"] & C^\infty \otimes \mathbb{S} \arrow[r, "\mathrm{div}"] & C^\infty \otimes \mathbb{V}
\end{tikzcd}
$$

Arnold, Winther: private communication

Homotopy operators on diagram

$$
\begin{tikzcd}[column sep=huge, row sep=large]
\begin{pmatrix} C^\infty\otimes\mathbb{V} \\ C^\infty\otimes\mathbb{V}\end{pmatrix} \arrow[r, "\begin{pmatrix} d & \\ & d\end{pmatrix}"] \arrow[d, "\Phi := \begin{pmatrix} I & K \\ 0 & I\end{pmatrix}"'] \arrow[ddd, dashed, red, bend right=60] &
\begin{pmatrix} C^\infty\otimes\mathbb{M} \\ C^\infty\otimes\mathbb{M}\end{pmatrix} \arrow[r, "\begin{pmatrix} d & \\ & d\end{pmatrix}"] \arrow[d] \arrow[l, dashed, red, bend right=40, "\begin{pmatrix} \mathfrak{p} & \\ & \mathfrak{p}\end{pmatrix}"'] &
\begin{pmatrix} C^\infty\otimes\mathbb{M} \\ C^\infty\otimes\mathbb{M}\end{pmatrix} \arrow[r, "\begin{pmatrix} d & \\ & d\end{pmatrix}"] \arrow[d] &
\begin{pmatrix} C^\infty\otimes\mathbb{V} \\ C^\infty\otimes\mathbb{V}\end{pmatrix} \arrow[d] \\
\begin{pmatrix} C^\infty\otimes\mathbb{V} \\ C^\infty\otimes\mathbb{V}\end{pmatrix} \arrow[r, "\begin{pmatrix} d & -S \\ 0 & d\end{pmatrix}"] \arrow[d, "\Pi^0"'] &
\begin{pmatrix} C^\infty\otimes\mathbb{M} \\ C^\infty\otimes\mathbb{M}\end{pmatrix} \arrow[r, "\begin{pmatrix} d & -S \\ 0 & d\end{pmatrix}"] \arrow[d, "\Pi^1"'] &
\begin{pmatrix} C^\infty\otimes\mathbb{M} \\ C^\infty\otimes\mathbb{M}\end{pmatrix} \arrow[r, "\begin{pmatrix} d & -S \\ 0 & d\end{pmatrix}"] \arrow[d, "\Pi^2"'] &
\begin{pmatrix} C^\infty\otimes\mathbb{V} \\ C^\infty\otimes\mathbb{V}\end{pmatrix} \arrow[d, "\Pi^3"'] \\
\Gamma^0 \arrow[r, "\begin{pmatrix} d & -S \\ 0 & d\end{pmatrix}"] \arrow[d, "\mathcal{I}"'] &
\Gamma^1 \arrow[r, "\begin{pmatrix} d & -S \\ 0 & d\end{pmatrix}"] \arrow[d, "\mathcal{I}"'] \arrow[uu, dashed, red, bend right=30, "\mathrm{id}"'] &
\Gamma^2 \arrow[r, "\begin{pmatrix} d & -S \\ 0 & d\end{pmatrix}"] \arrow[d, "\mathcal{I}"'] &
\Gamma^3 \arrow[d, "\mathcal{I}"'] \\
C^\infty\otimes\mathbb{V} \arrow[r, "\mathrm{def}"] &
C^\infty\otimes\mathbb{S} \arrow[r, "\mathrm{inc}"] \arrow[l, red, bend left=30, "\mathscr{P}"] \arrow[u, dashed, red, bend right=30, "\mathcal{I}^{-1}"'] &
C^\infty\otimes\mathbb{S} \arrow[r, "\mathrm{div}"] &
C^\infty\otimes\mathbb{V}
\end{tikzcd}
$$

Projections and lifting

Let $(W^\bullet, d^\bullet)$ be a subcomplex of $(V^\bullet, d^\bullet)$ and $\Pi^\bullet$ be cochain projections ($(\Pi)^2 = \Pi$, $d\Pi = \Pi d$).

$$\begin{tikzcd}
\cdots \arrow[r] & V^{i-1} \arrow[r, "d"] \arrow[d, "\Pi^{i-1}"'] & V^{i} \arrow[r, "d"] \arrow[d, "\Pi^{i}"'] & V^{i+1} \arrow[r] \arrow[d, "\Pi^{i+1}"'] & \cdots \\
\cdots \arrow[r] & W^{i-1} \arrow[r, "d"] & W^{i} \arrow[r, "d"] & W^{i+1} \arrow[r] & \cdots .
\end{tikzcd}$$

Lemma

If right inverse $\Pi_\dagger$ ($\Pi\Pi_\dagger = \mathrm{id}$) commutes with d, then

$$\tilde{\mathfrak{p}}^i := \Pi^{i-1}\mathfrak{p}^i\Pi^i_\dagger$$

defines $\tilde{\mathfrak{p}}^i : W^i \mapsto W^{i-1}$ for the subcomplex $(W^\bullet, d^\bullet)$ satisfying

$$d^{i-1}\tilde{\mathfrak{p}}^i + \tilde{\mathfrak{p}}^{i+1}d^i = \mathrm{id}.$$

Result

Theorem

$$\mathscr{P}_1(\omega) := \int_0^1 \omega_{tx} \cdot x dt + \int_0^1 (1-t)x \wedge (\nabla \times \omega_{tx}) \cdot x dt,$$

$$\mathscr{P}_2 : \mu \mapsto x \wedge \left(\int_0^1 t(1-t)\mu_{tx} dt \right) \wedge x,$$

$$\mathscr{P}_3 : \mu \mapsto \operatorname{sym} \left(\int_0^1 t^2 x \otimes \mu dt - \left(\int_0^1 t^2 (1-t) x \otimes \mu \wedge x dt \right) \times \nabla \right).$$

Then we have

$$\mathscr{P}_1(\operatorname{def} u) = u + \mathrm{RM}, \quad \forall u \in C^\infty \otimes \mathbb{V},$$

$$\mathscr{P}_2 \operatorname{inc} \mu + \operatorname{def} \mathscr{P}_1 \mu = \mu, \quad \forall \mu \in C^\infty \otimes \mathbb{S},$$

$$\mathscr{P}_3 \operatorname{div} \omega + \operatorname{inc} \mathscr{P}_2 \omega = \omega, \quad \forall \omega \in C^\infty \otimes \mathbb{S},$$

$$\operatorname{div} \mathscr{P}_3 v = v, \quad \forall v \in C^\infty \otimes \mathbb{V}.$$

- for $\mu \in C^\infty \otimes \mathbb{S}$ satisfying $\operatorname{inc}\mu = 0$, the Cesàro-Volterra path integral (1906, 1907)
$$\mu = \operatorname{def}\left(\mathscr{P}_1 \mu\right).$$
- complex property, polynomial-preserving property hold.

Koszul type operators

Define $\mathscr{K}_1^r : C^\infty \otimes \mathbb{S} \mapsto C^\infty \otimes \mathbb{V}$ by

$$\mathscr{K}_1^r : \omega \mapsto x \cdot \omega + \frac{1}{r+2} x \wedge (\nabla \times \omega) \cdot x, \quad \forall \omega \in C^\infty \otimes \mathbb{S},$$

and $\mathscr{K}_2^r : C^\infty \otimes \mathbb{S} \mapsto C^\infty \otimes \mathbb{S}$:

$$\mathscr{K}_2^r : u \mapsto x \wedge u \wedge x, \quad \forall u \in C^\infty \otimes \mathbb{S},$$

and define $\mathscr{K}_3^r : C^\infty \otimes \mathbb{V} \mapsto C^\infty \otimes \mathbb{S}$ by:

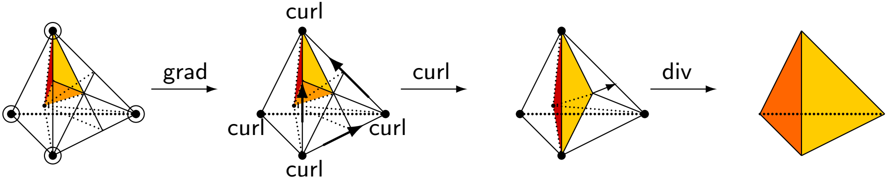
$$\mathscr{K}_3^r : v \mapsto \operatorname{sym}(x \otimes v) - \frac{1}{r+4} \operatorname{sym}\left((x \otimes v \wedge x) \times \nabla\right), \quad \forall v \in C^\infty \otimes \mathbb{V}.$$

- null-homotopy, polynomial preserving, Koszul type complex.
- duality:

$$\mathscr{K}_2^r u : v = u : \mathscr{K}_2^r v,$$

$$\int \mathscr{K}_1^{r+2} u : v = \int u : \mathscr{K}_3^r v.$$

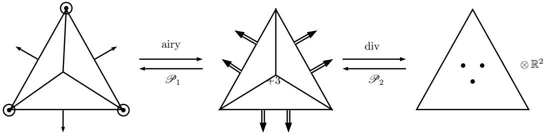
Stokes complex (de Rham version)



Low order Stokes complexes.

Christiansen, H.. *Generalized finite element systems for smooth differential forms and Stokes' problem*. Numerische Mathematik, May 2018.

Elasticity complex (2D, stress part)



$$0 \longrightarrow H^2 \xrightarrow{\text{curlcurl}} H(\text{div}; \mathbb{S}) \xrightarrow{\text{div}} L^2 \otimes \mathbb{V} \longrightarrow 0.$$

$$\mathcal{P}_{r+2}(T_{\mathrm{CT}}) \xrightarrow{\text{curlcurl}} \text{curlcurl}\,(\mathcal{P}_{r+2}(T_{\mathrm{CT}})) + \mathscr{P}_r^2(\mathcal{P}_r(T) \otimes \mathbb{V}) \xrightarrow{\text{div}} \mathcal{P}_r(T) \otimes \mathbb{V}.$$

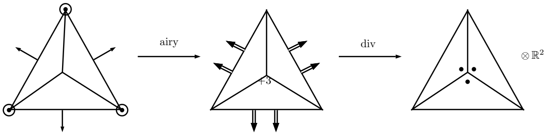
$$\mathscr{P}_1(V) = \int_0^1 (1-t) x^{\perp} \cdot V_{tx} \cdot x^{\perp}\, dt,$$

$$\mathscr{P}_2(u) = \text{sym}\left(\int_0^1 t u_{tx} \otimes x\, dt + \left(\int_0^1 t(t-1)(x^{\perp} \cdot u_{tx}) x\, dt \right) \times \nabla \right).$$

Arnold-Douglas-Gupta (Figure: $r = 1$.)

Christiansen, H.. *Finite Element System for vector bundles : elasticity and curvature*. arxiv.

Elasticity complex (2D, stress part)



$$0 \longrightarrow H^2 \xrightarrow{\text{curlcurl}} H(\text{div};\mathbb{S}) \xrightarrow{\text{div}} L^2 \otimes \mathbb{V} \longrightarrow 0.$$

$$0 \longrightarrow \mathcal{P}_{r+3}(T_h) \xrightarrow{\text{curlcurl}} \mathcal{P}_{r+1} \otimes \mathbb{S}(T_h) \xrightarrow{\text{div}} \mathcal{P}_r \otimes \mathbb{V}(T_h) \longrightarrow 0.$$

(**Figure**: $r = 0$.)

Christiansen, H.. *Finite Element System for vector bundles : elasticity and curvature*. arxiv.

More examples: 2D elasticity, strain part, curvature operator. Talk by Christiansen.

Discussions

- References
  - Poincaré path integrals for elasticity; Christiansen, H., Sande, *Journal de Mathématiques Pures et Appliquées*, 2019
  - Generalized finite element systems for smooth differential forms and Stokes' problem; Christiansen, H., Numerische Mathematik, 2018
  - Finite element systems for vector bundles: elasticity and curvature; Christiansen, H., *arXiv:1906.09128*
- elasticity (hess, divdiv...) Poincaré/Koszul: discrete complexes in higher dimensions? (ongoing for 3D)
- why complex property holds?
- a deeper understanding of BGG and Lie theory?
- BGG, defects, dislocations (Cosserat elasticity, microstructures, coupled stress, Riemannian and Cartan geometry, ..)?
- averaged integral operators and estimates as pseudo-differential operators?
- construction on manifolds (shells)? Cesàro-Volterra integral on surfaces (Ciarlet et al.)