Finite elements for curvature

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Outline

1. Elasticity complex and motivation
2. Finite element sequences
Elasticity complex and motivation

Finite element sequences
De Rham complexes

De Rham complex in 3D:

\[ \begin{array}{ccccccccc}
0 & \xrightarrow{\text{grad}} & L^2(\Omega) & \xrightarrow{\text{curl}} & L^2(\Omega) \otimes V & \xrightarrow{\text{div}} & L^2(\Omega) & \rightarrow & 0,
\end{array} \]

with the domain complex:

\[ \begin{array}{ccccccccc}
0 & \xrightarrow{\text{grad}} & H^1(\Omega) & \xrightarrow{\text{curl}} & H(\text{curl}, \Omega) & \xrightarrow{\text{div}} & H(\text{div}, \Omega) & \rightarrow & L^2(\Omega) & \rightarrow & 0.
\end{array} \]

- \( V \): vectors, \( M \): matrices, \( S \): symmetric matrices,
- \( H(\mathcal{D}, \Omega) := \{ u \in L^2 : \mathcal{D} u \in L^2 \} \),
- complex: \( \text{curl grad} = 0, \text{div curl} = 0 \),
- cohomology: \( \mathcal{N}(\text{curl})/\mathcal{R}(\text{grad}), \mathcal{N}(\text{div})/\mathcal{R}(\text{curl}) \).

The de Rham complex, as an example of more general Hilbert complexes, plays a vital role in the Finite Element Exterior Calculus (FEEC).
Elasticity complex (linearized Calabi complex)

\[
0 \rightarrow C^\infty \otimes V \xrightarrow{\text{def}} C^\infty \otimes \mathbb{S} \xrightarrow{\text{inc}} C^\infty \otimes \mathbb{S} \xrightarrow{\text{div}} C^\infty \otimes V \rightarrow 0
\]

\textit{displacement} \hspace{1cm} \textit{strain (metric)} \hspace{1cm} \textit{stress (curvature)} \hspace{1cm} \textit{force}

- $\nabla$: vectors, $\mathbb{S}$: symmetric matrices,
- linearized deformation \quad $\text{def } u := 1/2(\nabla u + u \nabla)$,
- linearized curvature \quad $\text{inc } v := \nabla \times v \times \nabla$,
- Saint-Venant compatibility condition: \quad $e = \text{def } u \Rightarrow \text{inc } e = 0$
- $\text{inc } \circ \text{def } = 0$, \quad $\text{div } \circ \text{inc } = 0$,
- classical elasticity
  - displacement formulation: displacement,
  - Hellinger-Reissner principle: stress+force,
  - intrinsic formulation: strain,
    \quad Ciarlet, Gratie, Mardare, 2009; Ciarlet, Ciarlet, 2008 etc.
- continuum description of defects (incompatibility theory)
  - elasto-plastic decomposition: $e = e^e + e^p$, $\text{inc } e^e = 0$,
  - Beltrami decomposition: $e = \text{def } w + \text{inc } v$. 

\[5/17\]
(A little bit) more on the continuous level

Continuous level has not been clarified yet...

Basically, we have all the analogous properties as the de Rham version. (Arnold, H., *Construction of Hilbert complexes*, in preparation.)

\[ \mathcal{H} \subset H^1(\mathbb{V}) \xrightarrow{\text{def}} H(\text{inc}; \mathbb{S}) \xrightarrow{\text{inc}} H(\text{div}; \mathbb{S}) \xrightarrow{\text{div}} L^2(\mathbb{V}) \longrightarrow 0. \]

- Cohomology is isomorphic to the de Rham cohomology \( \mathcal{H}_{dR} \otimes (\mathbb{V} \otimes \mathbb{V}) \),
- Operators have closed range,
- Poincaré type inequalities (→ Korn’s inequality),
- Hodge decomposition and well-posed Hodge Laplacian boundary value problems, (→ Beltrami type decomposition)
- Regular decomposition,
- Compactness property,
- Poincaré/Koszul operators (→ Cesàro-Volterra path integral),
Elasticity-electromagnetism analogue


Kröner [13] has developed a most useful analogy between the theory of internal stresses and strains as described in sections 2 to 6 and the theory of the magnetic field of distributions of stationary electric currents. Table 1 contains a list of the corresponding physical quantities, differential operators, and equations. We hope that this table is understandable without any further comments (see also the review article by de Wit [10]).

Table 1
Correspondences in elasticity and magnetism

<table>
<thead>
<tr>
<th>Elasticity</th>
<th>Magnetism</th>
</tr>
</thead>
<tbody>
<tr>
<td>vector quantity</td>
<td>scalar quantity</td>
</tr>
<tr>
<td>tensor rank two</td>
<td>vector</td>
</tr>
<tr>
<td>tensor rank four</td>
<td>tensor rank two</td>
</tr>
<tr>
<td>Div</td>
<td>div</td>
</tr>
<tr>
<td>Ink</td>
<td>curl</td>
</tr>
<tr>
<td>Div Ink ≡ 0</td>
<td>div curl ≡ 0</td>
</tr>
<tr>
<td>Def</td>
<td>grad</td>
</tr>
<tr>
<td>Ink Def ≡ 0</td>
<td>curl grad ≡ 0</td>
</tr>
<tr>
<td>Burgers vector ( \mathbf{b} )</td>
<td>current ( I )</td>
</tr>
<tr>
<td>incompatibility tensor ( \eta )</td>
<td>current density ( J )</td>
</tr>
<tr>
<td>strain tensor ( \varepsilon )</td>
<td>magnetic intensity ( H )</td>
</tr>
<tr>
<td>stress tensor ( \sigma )</td>
<td>magnetic induction ( B )</td>
</tr>
<tr>
<td>stress function tensor ( \chi, \chi' )</td>
<td>vector potential ( A )</td>
</tr>
<tr>
<td>elastic constants ( C ) (or ( G, K ))</td>
<td>permeability ( \mu )</td>
</tr>
<tr>
<td>displacement ( \mathbf{s} )</td>
<td>scalar potential ( \psi )</td>
</tr>
<tr>
<td>equation (3)</td>
<td>( \nabla^2 \mathbf{A} = - \mu \mathbf{J} )</td>
</tr>
<tr>
<td>equation (5)</td>
<td>( \text{div} \mathbf{A} = 0 )</td>
</tr>
<tr>
<td>equation (17)</td>
<td>( \mathbf{A} = \mu \int \int \frac{\mathbf{J}(\mathbf{r}')}{</td>
</tr>
<tr>
<td>equation (18)</td>
<td>( \mathbf{B} = \text{curl} \mathbf{A} )</td>
</tr>
<tr>
<td>equations (19), (19a)</td>
<td>( \text{div} \mathbf{B} = 0 )</td>
</tr>
<tr>
<td>equation (20)</td>
<td>( \text{curl} \mathbf{H} = \mathbf{J} )</td>
</tr>
<tr>
<td>equation (22)</td>
<td></td>
</tr>
</tbody>
</table>
a concrete model describing dislocation and defects (Amstutz, Van Goethem 2016):

\[
\begin{align*}
\text{inc inc } e &= f, \\
\text{div } e &= 0,
\end{align*}
\]

elasticity analogue of the Maxwell equations

\[
\begin{align*}
\text{curl curl } E &= g, \\
\text{div } E &= 0.
\end{align*}
\]

Question: how to discretize, even in 2D?

3D elasticity complex:

\[
\begin{align*}
\mathcal{R}M &\subset H^1 \otimes \mathbb{V} \xrightarrow{\text{def}} H(\text{inc}; \mathbb{S}) \xrightarrow{\text{inc}} H(\text{div}; \mathbb{S}) \xrightarrow{\text{div}} L^2 \otimes \mathbb{V} \rightarrow 0,
\end{align*}
\]

2D stress complex:

\[
\begin{align*}
\mathcal{P}_1 &\subset H^2 \xrightarrow{\text{airy}} H(\text{div}; \mathbb{S}) \xrightarrow{\text{div}} L^2 \otimes \mathbb{V} \rightarrow 0,
\end{align*}
\]

2D strain complex:

\[
\begin{align*}
\mathcal{R}M &\subset H^1 \otimes \mathbb{V} \xrightarrow{\text{def}} H(\text{rot rot}; \mathbb{S}) \xrightarrow{\text{rot rot}} L^2 \rightarrow 0.
\end{align*}
\]
Existing work

$H(\text{div, } \mathbb{S})-L^2(\nabla)$ pair:
- 2D macroelements: Johnson, Mercier 1978; Arnold, Douglas, Gupta 1984
- 2D and 3D : Arnold, Winther 2002; Arnold, Awanou, Winther 2008
- nD canonical construction: Hu, Zhang 2015
- other nonconforming methods, e.g., Pechstein, Schöberl 2012, TDNNS, $H(\text{div div; } \mathbb{S})-H(\text{curl})$

$H(\text{inc})$ element fitting into a sequence:
- Regge calculus from the finite element point of view (Christiansen 2011) piecewise $\mathcal{P}^0(\mathbb{S})$, tangential-tangential continuity, highly nonconforming,
- Li 2018 thesis: higher order Regge.

Goal of this work: first conforming 2D $H(\text{inc})$ element fitting in a sequence (conforming Regge type element)
1 Elasticity complex and motivation

2 Finite element sequences
Discrete stress complex: BGG approach


\[ \mathcal{P}_1 \subset H^2 \xrightarrow{\text{airy}} H(\text{div}; \mathbb{S}) \xrightarrow{\text{div}} L^2 \otimes \mathbb{V} \rightarrow 0, \]

\[ \mathbb{R} \subset H^2 \xrightarrow{\text{curl}} H^1 \otimes \mathbb{V} \xrightarrow{\text{div}} L^2 \rightarrow 0 \]

\[ \mathbb{V} \subset H^1 \otimes \mathbb{V} \xrightarrow{\text{curl}} H(\text{div}) \otimes \mathbb{V} \xrightarrow{\text{div}} L^2 \otimes \mathbb{V} \rightarrow 0 \]

(anti-)commuting diagram, \( dS = -Sd \), bijectivity, surjectivity
Output: Johnson-Mercier type elements.

Applying this BGG type diagram chase to other Stokes type complexes:

- other macroelements
Discrete stress complex: Poincaré operator approach

- Poincaré operators:
  - explicit potential, Poincaré lemma, canonical construction of FEs,
  - null-homotopy identity: $\mathcal{D}^{i-1} \mathcal{P}^i + \mathcal{P}^{i+1} \mathcal{D}^i = \text{id}$,
  - complex property: $\mathcal{P}^{i-1} \mathcal{P}^i = 0$,
  - polynomial-preserving property.

- Koszul operators: Poincaré acting on homogeneous polynomials, similar properties.

- for the stress complex (Christiansen, Hu, Sande 2019):

\[
\mathcal{K}_r^1 u = x^\perp \cdot u \cdot x^\perp,
\]

\[
(\mathcal{K}_r^2 u)(x) = \text{sym} \left( \frac{1}{r + 2} u \otimes x + \frac{1}{(r + 2)(r + 3)} \text{curl} \left( x^\perp \cdot ux \right) \right).
\]
Discrete strain complex: BGG approach

\[
\begin{align*}
\text{RM} & \xrightarrow{\subset} H^2 \otimes V \xrightarrow{\text{def}} H^1(\text{rot rot}; \mathbb{S}) \xrightarrow{\text{rot rot}} L^2 \xrightarrow{} 0. \\
V & \xrightarrow{\subset} H^2 \otimes V \xrightarrow{\text{grad}} H^1(\text{rot rot}; \mathbb{M}) \xrightarrow{\text{rot}} H(\text{rot}) \xrightarrow{} 0 \\
\mathbb{R} & \xrightarrow{\subset} H^1 \xrightarrow{\text{grad}} H(\text{rot}) \xrightarrow{\text{rot}} L^2 \xrightarrow{} 0
\end{align*}
\]
Output:

Equivalent edge DOFs: $\int_e \text{rot} \, u \cdot \tau \iff \int_e \partial_n (\tau \cdot u \cdot \tau)$ for $u \in C^0 \mathcal{P}^2$. 
Discrete strain complex: lower regularity

- Koszul operators for the strain complex:
  \[ \mathcal{K}_1^r (E) = \frac{1}{r+1} E \cdot x + \frac{1}{(r+1)(r+2)} x \wedge (\text{rot} E) \cdot x : C^\infty (\mathbb{S}) \mapsto C^\infty (\mathbb{V}), \]
  \[ \mathcal{K}_2^r (V) = \frac{1}{(r+2)(r+3)} x^\perp \otimes V \otimes x^\perp : C^\infty (\mathbb{S}) \mapsto C^\infty (\mathbb{S}). \]

- Strain complexes with lower regularity and fewer DOFs are possible:
  \[ \mathbb{V} \subset H^1 (\text{rot}) \xrightarrow{\text{def}} H (\text{rot}; \mathbb{S}) \cap H (\text{rot rot}; \mathbb{S}) \xrightarrow{\text{rot rot}} L^2 \longrightarrow 0 \]

- Can also be obtained by a diagram chase.
Conclusions

Reference:

- Finite element systems for vector bundles: elasticity and curvature; Christiansen, H., arXiv:1906.09128.
- Poincaré path integrals for elasticity; Christiansen, H., Sande, *Journal de Mathématiques Pures et Appliquées*, accepted, 2019

- discrete BGG diagram chase is based on several Stokes type complexes with various regularity, which is an important topic by itself,

- 3D is ongoing,

- comparison to discrete differential geometry and potential applications in defect theory, geometric problems (Ricci flows, Einstein equations etc.).