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What is a good basis? conditioning, sparsity, fast evaluation, rotational symmetry, etc.

... In this sense, preconditioning will always be an art rather than a science.


same remains true for bases.

This talk: conditioning for high order finite element bases (dependence on degree).
1. Condition number: a revisit

2. High order finite elements
finite element methods: with some bases \( \{ \phi_i \}_{i=1}^N \),

\[
a(u, v) := \int_{\Omega} \nabla u \cdot \nabla v \, dx, \quad (u, v) := \int_{\Omega} u \cdot v \, dx,
\]

stiffness matrix \( (A_h)_{ij} := a(\phi_i, \phi_j) \), mass matrix \( (M_h)_{ij} := (\phi_i, \phi_j) \).

condition number \( \kappa(A_h), \kappa(M_h) \): depend on the basis

design bases: a lot of efforts

Fuentes, Keith, Demkowicz, Nagaraj, 2015; Beuchler, Pillwein, Schöberl, Zaglmayr 2012; Szabo, Babuška 1991; Karniadakis, Sherwin 2013; Bonazzoli, Gaburro, Dolean, Rapetti 2014; Ainsworth, Coyle, 2004; Dubiner 1991 etc.

Question: Which condition number to optimize? How?
e.g., convection-diffusion equation

\[
 u_t + (\beta(x) \cdot \nabla) u - \epsilon \Delta u = f.
\]

mass varying coefficients stiffness
Our choice: $L^2$ (mass) condition number

Reasons:

- "basis condition number".

representation $\tau_h : \mathbb{R}^n \mapsto V_h$, $\tau_h(c) = \sum_j c_j \phi_j$,

operator $A_h : V_h \rightarrow V_h^*$, matrix representation $A_h := \tau_h^* A_h \tau_h : \mathbb{R}^n \rightarrow \mathbb{R}^n$

\[
\begin{align*}
\mathbb{R}^n & \xrightarrow{A_h} \mathbb{R}^n \\
& \xrightarrow{\tau_h} V_h \\
V_h & \xrightarrow{A_h} V_h^*
\end{align*}
\]

Riesz operator $I_h : V_h^* \mapsto V_h$, mass matrix $M_h = \tau_h^* I_h^{-1} \tau_h$,

\[A_h = M_h (\tau_h^{-1} I_h A_h \tau_h), \quad \Rightarrow \quad \kappa(A_h) \leq \kappa(I_h A_h) \cdot \kappa(M_h),\]

$L^2$ condition number controls others: $\kappa(I_h A_h) = \kappa(M_h^{-1} A_h) \leq Cr^4 $

Poincaré + inverse inequalities

$\|\nabla u_h\| \leq Ch^{-1} \|u_h\|, \quad \|\nabla u_h\| \leq Cr^2 \|u_h\|. \text{ (sharp bound: Bernardi, Maday 1997)}$

resemblance of $h$-method: in a position to handle stiffness matrix by preconditioners
1 Condition number: a revisit

2 High order finite elements
High order finite elements: continuous piecewise polynomial

\[ V_r := \{ u_h \in C^0(\Omega) : u_h|_T \in P_r(T), \forall T \in T_h \}. \]

- natural bases (1D): vertex modes + interior modes/bubbles

\[ V_r = \sum_{v \in \mathcal{V}} V_v \oplus \sum_{e \in \mathcal{E}} V_e \]

Example: \( r = 3. \)
Local orthogonality is not enough for well conditioning

\[ V_r = \sum_{v \in V} V_v \oplus \sum_{e \in \mathcal{E}} V_e \]

hat function \(\oplus\) orthogonal bubbles

condition numbers grow with the polynomial degree \(r\).

**Reason:** vertex hat functions interfere with bubbles

![Figure: DoF decomposition: interior modes of Legendre polynomials, \(\log(\kappa)\).](image-url)
Quick overview of our solution: stable decomposition of $V_r$ by introducing redundancy

$$V_r = \sum_{v \in \mathcal{V}} \tilde{V}_v + \sum_{e \in \mathcal{E}} V_e$$

more hat functions + orthogonal bubbles

$L^2$ frame condition numbers ($\lambda_{\text{max}}/\lambda_{\text{min},+}$) remains constant for all $r$:
(reminder: frame = bases with redundancy)
More details in $nD$: well conditioning $= \text{stable decomposition} + \text{local orthogonality}$

- notation: $f \in \Delta$: vertex, edge, face etc., $\Omega_f$: patch associated with $f$.

- tool: bubble transform (Falk, Winther, 2016)
  stable decomposition of $H^1$ into local patches, $u = \sum_{f \in \Delta} B_f u$,
  $a \|u\|^2 \leq \sum_{f \in \Delta} \|B_f u\|^2 \leq b \|u\|^2, \quad \forall u \in H^1(\Omega)$.

- bubble transform on finite element spaces:
  \[
  V_r = \sum_{f \in \Delta} B_f V_r =: \sum_{f \in \Delta} V_f,
  \]

  where $V_f \subset \mathcal{P}_r(\Omega_f)$ is a pull back of polynomial spaces defined on reference elements to $\Omega_f$. not a direct sum
Theorem

Assume that there exists $B_f : V_r \mapsto V_f$, and (stable decomposition)

$$a\|u\|^2 \leq \sum_{f \in \Delta} \|B_fu\|^2 \leq b\|u\|^2,$$

and (local basis $\phi_{f,k}$ in $V_f$)

$$\alpha_f \sum_k c_{f,k}^2 \leq \left\| \sum_k c_{f,k} \phi_{f,k} \right\|^2 \leq \beta_f \sum_k c_{f,k}^2, \quad \forall c_{f,k}.$$

Define the local condition number $\kappa_f = \beta_f / \alpha_f$. We have

$$\kappa \leq (a^{-1}b) \left( \max_{f \in \Delta} \kappa_f \right) \cdot \max_{f,g \in \Delta} \frac{\alpha_f}{\alpha_g},$$

where $\kappa$ is the (global) frame condition number.

frame condition number $\kappa := \lambda_{\max} / \lambda_{\min,+}$.
Scaling of local blocks

\[
\max_{f, g \in \Delta} \frac{\alpha_f}{\alpha_g} : \text{scaling of local blocks}
\]

Example:

\[
A = \begin{pmatrix} I & 0 \\ 0 & \epsilon I \end{pmatrix},
\]

condition number of each block is 1, but \( \max \frac{\alpha_k}{\alpha_j} = \epsilon^{-1} \).
Local orthogonal bases: Jacobi polynomials on simplices

- mutually orthogonal (Jacobi) polynomials on \((m + 1)\)-simplexes with weight \(w_m\):

\[
J_s(\lambda) := c_s^{-1} \prod_{j=0}^{m} \left(1 - \sum_{i=0}^{j-1} \lambda_i \right)^{s_j} J_{a_j,2}^{s_j} \left(\frac{2\lambda_j}{1 - \sum_{i=0}^{j-1} \lambda_i} - 1\right), \quad |s| \leq s,
\]

where \(s = (s_0, \ldots, s_m)\) is a multi-index,

\[
a_j = 2 \sum_{i=j+1}^{m} s_i + d + 2m - 3j - 1,
\]

and

\[
c_s^{-2} = \prod_{j=0}^{m} 2^{a_j+3} = \prod_{j=0}^{m} 2^{2 \sum_{i=j+1}^{m} s_i+d+2m-3j+2}.
\]

(c.f. Dunkl, Xu 2014.)

- Duffy transform:

Singularity and non-symmetry arise (figure from Mengaldo, Grazia, Moxey, Vincent, Sherwin 2015)
2D tests: robust with mesh distortion

Table: Results for Test 1.

<table>
<thead>
<tr>
<th>r</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>9</th>
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</thead>
<tbody>
<tr>
<td>$\lambda_{\text{max}}$</td>
<td>6.623</td>
<td>6.819</td>
<td>6.893</td>
<td>6.930</td>
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<tr>
<td>$\lambda_{\text{min,+}}$</td>
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<td>0.457</td>
<td>0.472</td>
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<tr>
<td>Dim. of frame</td>
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<td>98</td>
<td>199</td>
<td>336</td>
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<tr>
<td>Rank of frame</td>
<td>19</td>
<td>46</td>
<td>85</td>
<td>136</td>
</tr>
</tbody>
</table>

Table: Results for Test 2.

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<tbody>
<tr>
<td>$\lambda_{\text{max}}$</td>
<td>6.624</td>
<td>6.819</td>
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<td>6.930</td>
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<tr>
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</tbody>
</table>
**Figure**: Pattern of mass matrix, Test 1, $r = 3$.

**Figure**: Pattern of mass matrix, Test 1, $r = 9$. 
Solving semi-definite systems

- most iterative methods work for semi-definite systems

connections to iterative methods:

Jacobi/Gauss-Seidel on frames $\iff$ subspace correction iterations (MG, DDM etc.),

Example of multigrid:

$p$- preconditioning with a similar decomposition:

take-home message:
- consider $L^2$ (mass) condition number,
- redundancy + local orthogonality $\Rightarrow$ well-conditioning.

further directions:
- more general local shape functions (other than polynomials)
  (bubble transform still works)
- electromagnetism, fluid, elasticity... : curl/div problems
- $p$- preconditioning for the local problem (spectral methods).

Reference: