

Structure-preserving finite element methods for incompressible MHD systems

Kaibo Hu

University of Oxford

based on joint papers with Young-Ju Lee (Texas), Jinchao Xu (Penn State); and Patrick E. Farrell (Oxford), Fabian Laakmann (Oxford)

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Outline

- 1 MHD and conserved quantities
- 2 Structure-preserving discretisation
- 3 Numerical experiments

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Magnetohydrodynamics (MHD): An incompressible model

$$\partial_t \mathbf{u} - \mathbf{u} \times (\nabla \times \mathbf{u}) - R_e^{-1} \Delta \mathbf{u} - \textcolor{red}{S} \mathbf{j} \times \mathbf{B} + \nabla P = \mathbf{f} \quad \text{Newton's law,}$$

$$\mathbf{j} - \nabla \times \mathbf{B} = 0 \quad \text{Ampere's law,}$$

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0 \quad \text{Faraday's law,}$$

$$R_m^{-1} \mathbf{j} - (\mathbf{E} + \mathbf{u} \times \mathbf{B}) = 0 \quad \text{Ohm's law,}$$

$$\nabla \cdot \mathbf{B} = 0 \quad \text{Gauss law,}$$

$$\nabla \cdot \mathbf{u} = 0,$$

initial conditions $\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x})$, $\mathbf{B}(\mathbf{x}, 0) = \mathbf{B}_0(\mathbf{x})$,

boundary conditions on $\partial\Omega$: $\mathbf{u} = 0$, $\mathbf{B} \cdot \mathbf{n} = 0$, $\mathbf{E} \times \mathbf{n} = 0$.

Three nonlinear terms:

fluid advection $-\mathbf{u} \times (\nabla \times \mathbf{u})$ (in the vorticity form)

Lorentz force $-\textcolor{red}{S} \mathbf{j} \times \mathbf{B}$

magnetic advection $-\nabla \times (\mathbf{u} \times \mathbf{B})$

Ideal MHD: $R_e^{-1} = R_m^{-1} = 0$.

Energy structures of MHD

Energy:

$$\mathcal{E} := \|\mathbf{u}\|^2 + S\|\mathbf{B}\|^2.$$

$$\frac{1}{2} \frac{d\mathcal{E}}{dt} = -R_e^{-1} \|\nabla \mathbf{u}\|^2 - SR_m^{-1} \|\mathbf{j}\|^2 + (\mathbf{f}, \mathbf{u}),$$

and hence

$$\begin{aligned} \max_{0 \leq t \leq T} \mathcal{E}(t) \leq & -R_e^{-1} \int_0^T \|\nabla \mathbf{u}\|^2 d\tau - 2SR_m^{-1} \int_0^T \|\mathbf{j}\|^2 d\tau \\ & + \|\mathbf{u}_0\|^2 + S\|\mathbf{B}_0\|^2 + R_e \int_0^T \|\mathbf{f}\|_{-1}^2 d\tau. \end{aligned}$$

key structure: fluid advection cancels itself, Lorentz force and magnetic advection cancel each other

Test $\mathbf{u}$ to the momentum eqn: $(\mathbf{j} \times \mathbf{B}, \mathbf{u}) = ((\nabla \times \mathbf{B}) \times \mathbf{B}, \mathbf{u})$; test $\mathbf{B}$ to the Faraday's law: $(\nabla \times (\mathbf{u} \times \mathbf{B}), \mathbf{B})$. Reflecting symmetry in PDEs, useful for solvers.

Magnetic Gauss law: constraint propagation

Magnetic Gauss's law: $\nabla \cdot \mathbf{B} = 0, \quad \forall t \geq 0.$

Constraint propagation:

$$\mathbf{B}_t + \nabla \times \mathbf{E} = 0 \quad \Longrightarrow \quad \frac{d}{dt}(\nabla \cdot \mathbf{B}) = 0,$$

$$\nabla \cdot \mathbf{B}_0 = 0 \quad \Longrightarrow \quad \nabla \cdot \mathbf{B}(\mathbf{x}, t) = 0, \quad \forall t > 0.$$

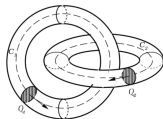
Divergence-constraint automatically satisfied by evolution, not an independent equation.

Helicity: conserved in ideal MHD

Magnetic helicity: for magnetic potential $\mathbf{A}$ satisfying $\nabla \times \mathbf{A} = \mathbf{B}$,

$$\text{magnetic helicity } \mathcal{H}_m := \int_{\Omega} \mathbf{A} \cdot \mathbf{B} \, dx$$

characterizing linking/knottedness of $\mathbf{B}$. *magnetic advection cancels itself*



Arnold, Khesin, *Topological methods in hydrodynamics*, 1999

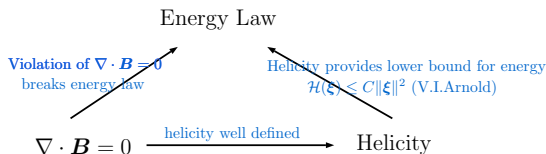
Idea started from Helmholtz and Kelvin. MHD: Woltjer's invariant, ideal fluid: Moffatt (giving the name).

Cross helicity:

$$\text{cross helicity } \mathcal{H}_c := \int_{\Omega} \mathbf{u} \cdot \mathbf{B} \, dx$$

Lorentz force cancels itself, fluid and magnetic advection cancel each other

Structure-preserving discretisation



Important to preserve structures precisely:

- **energy conservation**: e.g., Armero, Simo 1996
- **$\nabla \cdot \mathbf{B} = 0$** : e.g., Brackbill, Barnes 1980
- **helicity conservation**: less attention

Liu, Wang 2004 (axisymmetric MHD flow, finite difference methods), Rebholz 2007 (NS eqn, FEM), Kraus, Maj 2017 (DEC, variational integrator)

More recent works: Sullivan (2018, combinatorial topology), Gawlik, Gay-Balmaz (2021), Zhang, Palha, Gerritsma, Rebholz (2022), Wimmer, Tang (2022)

Role of helicity-preservation

Physical fidelity: MHD dynamo, reconnection, dynamical events in turbulence etc.

2D: numerical oscillation

$$\|\nabla \mathbf{B}\|^2 \leq C(\underbrace{\|\nabla \cdot \mathbf{B}\|^2}_{\text{divergence-free condition: } \nabla \cdot \mathbf{B} = 0} + \underbrace{\|\nabla \times \mathbf{B}\|^2}_{\text{enstrophy: 'helicity' in even dimension}})$$

Numerical oscillations are unlikely, if both $\nabla \times \mathbf{B}$ and $\nabla \cdot \mathbf{B}$ are controlled.

Goal: Finite element discretization naturally preserving all these quantities.

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Cohomology and MHD structure-preservation: motivation

Continuous level:

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0 \implies \partial_t(\nabla \cdot \mathbf{B}) = 0.$$

Typical Galerkin schemes:

Find $\mathbf{E}_h \in Y_h$, $\mathbf{B}_h \in Z_h$, s.t.

$$(\partial_t \mathbf{B}_h, \mathbf{C}_h) + (\nabla \times \mathbf{E}_h, \mathbf{C}_h) = 0, \quad \forall \mathbf{C}_h \in Z_h.$$

Consequence: $\partial_t \mathbf{B}_h + \mathbb{P}_{Y_h} \nabla \times \mathbf{E}_h = 0$, hence $\partial_t(\nabla \cdot \mathbf{B}_h) = -\nabla \cdot \mathbb{P}_{Y_h} \nabla \times \mathbf{E}_h$.

$\mathbb{P}_{Y_h}$: L^2 projection from $\nabla \times Y_h$ to Z_h .

$\mathbb{P}_{Y_h} = I$, hence $\partial_t(\nabla \cdot \mathbf{B}_h) = 0$, if $\nabla \times Y_h \subset Z_h$.

Relations like $\nabla \times Y_h \subset Z_h$ are studied in *homological algebra*.

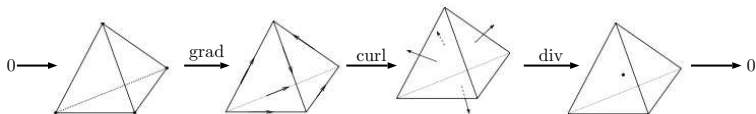
de Rham complex in $\mathbb{R}^3$

$$0 \longrightarrow H_0(\text{grad}) \xrightarrow{\text{grad}} H_0(\text{curl}) \xrightarrow{\text{curl}} H_0(\text{div}) \xrightarrow{\text{div}} L^2 \longrightarrow 0.$$

where $H_0(D) := \{u \in L^2 : Du \in L^2, \text{tr } u = 0 \text{ on } \partial\Omega\}$.

Complex: $\text{curl} \circ \text{grad} = 0$, $\text{div} \circ \text{curl} = 0$.

Finite element version:



$$0 \longrightarrow H_0^h(\text{grad}) \xrightarrow{\text{grad}} H_0^h(\text{curl}) \xrightarrow{\text{curl}} H_0^h(\text{div}) \xrightarrow{\text{div}} L_{0,h}^2 \longrightarrow 0.$$

Raviart-Thomas, Nédélec in numerical analysis, Whitney forms (1957).

discrete differential forms (Bossavit 1988, Hiptmair 1999), Finite Element Exterior Calculus (FEEC, Arnold, Falk, Winther 2006).

The key: choosing FE spaces in a de Rham complex

Linear invariant $\nabla \cdot \mathbf{B} = 0$: choosing $\mathbf{B}$ in $H_0^h(\text{div})$ and $\mathbf{E} \in H_0^h(\text{curl})$.

Complex: $\text{curl } H_0^h(\text{curl}) \subset H_0^h(\text{div})$

Quadratic invariant: magnetic helicity

Continuous level:

$$\int \nabla \times (\mathbf{u} \times \mathbf{B}) \cdot \mathbf{A} = \int (\mathbf{u} \times \mathbf{B}) \cdot \mathbf{B} = 0.$$

Discrete level (typical mixed finite element scheme):

$$\int \nabla \times \mathbb{Q}_h^{\text{curl}}(\mathbf{u} \times \mathbf{B}) \cdot \mathbf{A} = \int \mathbb{Q}_h^{\text{curl}}(\mathbf{u} \times \mathbf{B}) \cdot \nabla \times \mathbf{A} = \int (\mathbf{u} \times \mathbf{B}) \cdot \mathbb{Q}_h^{\text{curl}} \mathbf{B} \neq 0,$$

where $\mathbb{Q}_h^{\text{curl}}$ is the L^2 projection to $H_0^h(\text{curl})$.

Fix: introduce $\mathbf{H} = \mathbb{Q}_h^{\text{curl}} \mathbf{B}$, use $\nabla \times \mathbb{Q}_h^{\text{curl}}(\mathbf{u} \times \mathbf{H})$ in the scheme.

$$\int (\mathbf{u} \times \mathbb{Q}_h^{\text{curl}} \mathbf{B}) \cdot \mathbb{Q}_h^{\text{curl}} \mathbf{B} = 0.$$

Cross helicity: similar, introduce $\boldsymbol{\omega} := \mathbb{Q}_h^{\text{curl}} \nabla \times \mathbf{u}$ and modify the scheme accordingly.

$$\mathbf{X} := H_0(\text{curl}) \times H_0(\text{div}) \times H_0(\text{grad}), \quad \mathbf{X}_h := H_0^h(\text{curl}) \times H_0^h(\text{div}) \times H_0^h(\text{grad}).$$

Discrete variational form:

Find $(\mathbf{u}_h, \mathbf{B}_h, P_h) \in \mathbf{X}_h$, such that for all $(\mathbf{v}_h, \mathbf{C}_h, Q_h) \in \mathbf{X}_h$,

$$\begin{aligned} (D_t \mathbf{u}_h, \mathbf{v}_h) + (\mathbb{Q}_h^{\text{curl}} [\nabla \times \mathbf{u}_h] \times \mathbf{u}_h, \mathbf{v}_h) - S((\nabla_h \times \mathbf{B}_h) \times \mathbb{Q}_h^{\text{curl}} \mathbf{B}_h, \mathbf{v}_h) + (\nabla P_h, \mathbf{v}_h) &= 0, \\ (\mathbf{u}_h, \nabla Q_h) &= 0, \\ (D_t \mathbf{B}_h, \mathbf{C}_h) - (\nabla \times \mathbb{Q}_h^{\text{curl}} [\mathbf{u}_h \times \mathbb{Q}_h^{\text{curl}} \mathbf{B}_h], \mathbf{C}_h) &= 0, \end{aligned}$$

where $\nabla_h \times : H_0^h(\text{div}) \rightarrow H_0^h(\text{curl})$ is the L^2 adjoint of $\text{curl} : H_0^h(\text{curl}) \rightarrow H_0^h(\text{div})$:

$$(\nabla_h \times \mathbf{B}_h, \mathbf{k}_h) = (\mathbf{B}_h, \nabla \times \mathbf{k}_h), \quad \forall \mathbf{k}_h \in H_0^h(\text{curl}).$$

$D_t \mathbf{u} = (\mathbf{u}^{\text{new}} - \mathbf{u}^{\text{old}})/\Delta t$, $D_t \mathbf{B} = (\mathbf{B}^{\text{new}} - \mathbf{B}^{\text{old}})/\Delta t$ and other variables are average of new and old values (*Crank-Nicolson time stepping*).

Magnetic and cross helicity, energy and $\nabla \cdot \mathbf{B} = 0$ preserved.

Implementation of the projections: (equivalent) mixed FEM scheme

Find $(\mathbf{u}, \boldsymbol{\omega}, \mathbf{j}, \mathbf{E}, \mathbf{H}, \mathbf{B}, p) \in [H_0^h(\text{curl}, \Omega)]^5 \times H_0^h(\text{div}, \Omega) \times H_0^h(\text{grad})$ such that

$$(D_t \mathbf{u}, \mathbf{v}) - (\mathbf{u} \times \boldsymbol{\omega}, \mathbf{v}) + (\nabla p, \mathbf{v}) - S(\mathbf{j} \times \mathbf{H}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}),$$

$$(\boldsymbol{\omega}, \boldsymbol{\mu}) - (\nabla \times \mathbf{u}, \boldsymbol{\mu}) = 0,$$

$$(\mathbf{u}, \nabla q) = 0,$$

$$(D_t \mathbf{B}, \mathbf{C}) + (\nabla \times \mathbf{E}, \mathbf{C}) = 0,$$

$$(\mathbf{j}, \mathbf{k}) - (\mathbf{B}, \nabla \times \mathbf{k}) = 0,$$

$$(\mathbf{E} + \mathbf{u} \times \mathbf{H}, \mathbf{G}) = 0,$$

$$(\mathbf{B}, \mathbf{F}) - (\mathbf{H}, \mathbf{F}) = 0,$$

Equations can be interpreted as projections:

$$\mathbf{E} = -\mathbb{Q}_h^{\text{curl}}(\mathbf{u} \times \mathbf{H}),$$

$$\boldsymbol{\omega} = \mathbb{Q}_h^{\text{curl}}(\nabla \times \mathbf{u}),$$

$$\mathbf{j} = \nabla_h \times \mathbf{B}, \quad \mathbf{H} = \mathbb{Q}_h^{\text{curl}} \mathbf{B}.$$

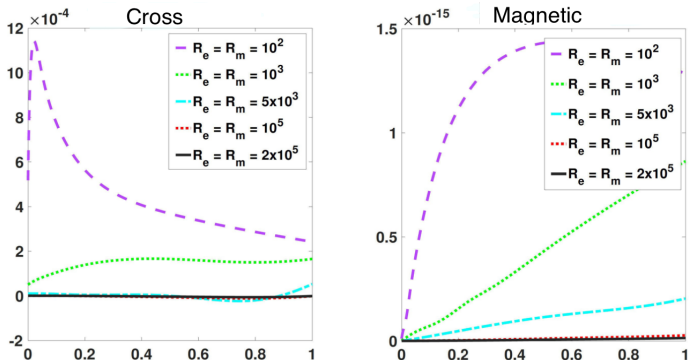
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Convergence test

h	$\ \mathbf{B} - \mathbf{B}_h\ _0$	order	$\ \mathbf{u} - \mathbf{u}_h\ _0$	order	$\ p - p_h\ _1$	order
2^{-2}	1.60E-3	x	4.15E-4	x	2.15E-4	x
2^{-3}	7.80E-4	1.04	2.18E-4	0.93	1.24E-4	0.79
2^{-4}	3.40E-4	1.20	1.05E-4	1.05	6.44E-5	0.95
2^{-5}	1.63E-4	1.06	5.30E-5	0.99	3.25E-5	0.99

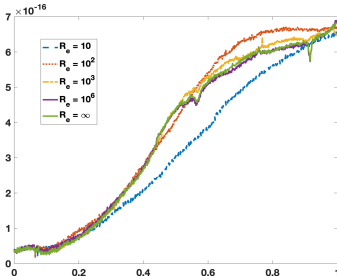
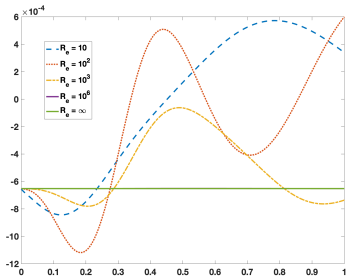
Convergence results for the MHD system. The error is computed at the time level $T = 1$ with the Crank-Nicolson time stepping with $\Delta t = 0.01$. $R_e = R_m = 10^4$.

Cross and magnetic helicity: various choices of R_e and R_m



Various choices of R_e and R_m . Left: cross helicity. Right: magnetic helicity.

Helicity nearly conserved for large R_e and R_m .



$R_m = 10^7$, various R_e . Left: cross helicity. Right: magnetic helicity.

Magnetic helicity conservation is independent of u .

Extension to Hall MHD equations

$$R_m^{-1} \mathbf{j} - (\mathbf{E} + \mathbf{u} \times \mathbf{B} - R_H \mathbf{j} \times \mathbf{B}) = 0.$$

Structures: the Hall term significantly changes the structure of PDEs

$\mathbf{j} \times \mathbf{B} = (\nabla \times \mathbf{B}) \times \mathbf{B} = (\nabla \times \nabla \times \mathbf{A}) \times (\nabla \times \mathbf{A})$ is a high order term.

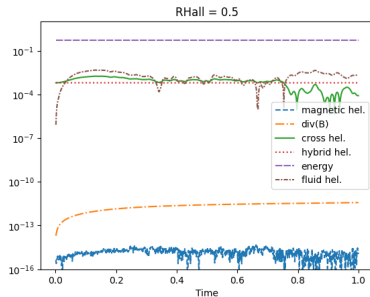
Cross helicity is not conserved, while the hybrid helicity is

$$H_H := \int_{\Omega} (\mathbf{A} + \alpha \mathbf{u}) \cdot (\mathbf{B} + \beta \nabla \times \mathbf{u}) \, dx$$

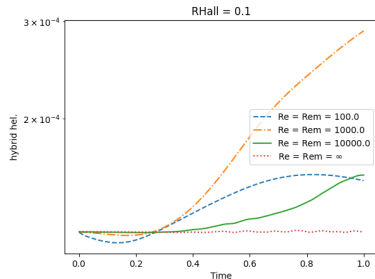
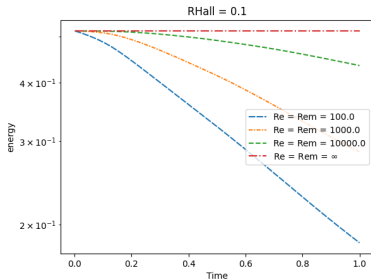
for certain α and β .

Discretization: extensions of mixed formulations, with $R_H(\nabla \times \mathbb{Q}_h^{\text{curl}}[\mathbf{j}_h \times \mathbb{Q}_h^{\text{curl}} \mathbf{B}_h], \mathbf{C}_h)$.

Magnetic and hybrid helicity, energy and $\nabla \cdot \mathbf{B} = 0$ preserved.



Conserved quantities in the ideal limit.



Energy (left) and hybrid helicity (right) for different values of R_e and R_m .

Conclusions

Conclusions and remarks:

- Preserving conserved quantities using finite element [de Rham complexes](#),
- Functional analytic and algebraic [preconditioners](#) can be constructed and analyzed based on mathematical structures,
- Further directions: turbulence.

References:

- *Helicity-conservative finite element discretization for incompressible MHD systems*, KH, Y.-J. Lee, J. Xu; Journal of Computational Physics, 2021.
- *Structure-preserving and helicity-conserving finite element approximations and preconditioning for the Hall MHD equations*, F. Laakmann, KH, P. E. Farrell (2022) arXiv:2202.11586