BGG machinery and bounded Poincaré operators

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Three complexes

de Rham complex topology, curl, Poincaré inequality

$$0 \longrightarrow C^\infty \xrightarrow{\text{grad}} C^\infty \otimes \mathbb{R}^3 \xrightarrow{\text{curl}} C^\infty \otimes \mathbb{R}^3 \xrightarrow{\text{div}} C^\infty \longrightarrow 0.$$

elasticity complex Riemannian geometry, curl $\circ \mathrm{T} \circ$ curl, Korn inequality

displacement formulation

Kröner's continuum description of dislocations/defects, internal stress

$$0 \longrightarrow C^\infty \otimes \mathbb{V} \xrightarrow{\text{def}} C^\infty \otimes \mathbb{S} \xrightarrow{\text{inc}} C^\infty \otimes \mathbb{S} \xrightarrow{\text{div}} C^\infty \otimes \mathbb{V} \longrightarrow 0$$

displacement *strain (metric)* *stress (curvature)* *force*

intrinsic elasticity (Ciarlet et al.)

Hellinger-Reissner principle of elasticity

$\mathbb{V}$: vectors, $\mathbb{S}$: symmetric matrices conformal complex conformal geometry, $\operatorname{curl} \circ S^{-1} \circ \operatorname{curl} \circ S^{-1} \circ \operatorname{curl}$, conformal Korn inequality

ker of dev def: conformal Killing v.f.

Cotton-York: flatness in conformal geometry

$$0 \longrightarrow H^s(\Omega)\otimes\mathbb{V} \xrightarrow{\operatorname{dev}\operatorname{def}} H^{s-1}(\Omega)\otimes(\mathbb{S}\cap\mathbb{T}) \xrightarrow{\operatorname{Cott}} H^{s-4}(\Omega)\otimes(\mathbb{S}\cap\mathbb{T}) \xrightarrow{\operatorname{div}} H^{s-5}(\Omega)\otimes\mathbb{V} \longrightarrow 0$$

gravitational wave variable: transverse-traceless (TT) gauge
(= symmetric, trace-free, div-free)

stress like variable def$u$ in NS
(Gopalakrishnan, Lederer, Schöberl, 2019)

$\mathbb{T}$: trace-free matrices

$\operatorname{dev} w := w - \frac{1}{n}\operatorname{tr}(w)I, \quad \operatorname{cott} g := \operatorname{curl} S^{-1} \operatorname{curl} S^{-1} \operatorname{curl}, \quad \operatorname{div} v := \nabla\cdot v$
$Su := u^T - \operatorname{tr}(u)I$

- NS: stress-like variable $\sigma := \operatorname{def}(u)$ (Gopalakrishnan, Lederer, Schöberl 2020)
- transverse-traceless (TT) gauge of gravitational waves
- maximal slicing $K = 0$ (for constructing black hole initial data)
- gravity shielding, linearized Einstein constraint equations (Beig, Chruściel 2020)

Conformal complex: 2-rows approach (Arnold,KH 2021)

either eliminate trace from symmetric matrices:

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & H^q \otimes \mathbb{V} & \xrightarrow{\text{dev grad}} & H^{q-1} \otimes \mathbb{T} & \xrightarrow{\text{sym curl}} & H^{q-2} \otimes \mathbb{S} & \xrightarrow{\text{div div}} & H^{q-4} & \longrightarrow & 0 \\
 & & & \nearrow^{-\text{mskw}} & & \nearrow^{\mathcal{S}} & & \nearrow^{\text{tr}} & & & \\
0 & \longrightarrow & H^{q-1} \otimes \mathbb{V} & \xrightarrow{\text{def}} & H^{q-2} \otimes \mathbb{S} & \xrightarrow{\text{inc}} & H^{q-4} \otimes \mathbb{S} & \xrightarrow{\text{div}} & H^{q-5} \otimes \mathbb{V} & \longrightarrow & 0
\end{array}
$$

or eliminate skew-symmetric part from trace-free matrices:

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & H^{q-1} \otimes \mathbb{V} & \xrightarrow{\text{def}} & H^{q-2} \otimes \mathbb{S} & \xrightarrow{\text{inc}} & H^{q-4} \otimes \mathbb{S} & \xrightarrow{\text{div}} & H^{q-5} \otimes \mathbb{V} & \longrightarrow & 0 \\
 & & & \nearrow^{\iota} & & \nearrow^{\mathcal{S}} & & \nearrow^{2\,\text{vskw}} & & & \\
0 & \longrightarrow & H^{q-2} & \xrightarrow{\text{hess}} & H^{q-4} \otimes \mathbb{S} & \xrightarrow{\text{curl}} & H^{q-5} \otimes \mathbb{T} & \xrightarrow{\text{div}} & H^{q-6} \otimes \mathbb{V} & \longrightarrow & 0.
\end{array}
$$

Remaining question: cohomology not clear. [no K operators, such that $S = DK - KD$.]

BGG in a more general setting (Čap,KH 2022)

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & Z^{0,0} & \xrightarrow{d^{0,0}} & Z^{1,0} & \xrightarrow{d^{1,0}} & \cdots & \xrightarrow{d^{n-1,0}} & Z^{n,0} & \longrightarrow & 0 \\
 & & K\uparrow & \nearrow^{S^{0,1}} & K\uparrow & \nearrow^{S^{1,1}} & K\uparrow & \nearrow^{S^{n-1,1}} & K\uparrow & & \\
0 & \longrightarrow & Z^{0,1} & \xrightarrow{d^{0,1}} & Z^{1,1} & \xrightarrow{d^{1,1}} & \cdots & \xrightarrow{d^{n-1,1}} & Z^{n,1} & \longrightarrow & 0 \\
 & & K\uparrow & \nearrow^{S^{0,2}} & K\uparrow & \nearrow^{S^{1,2}} & K\uparrow & \nearrow^{S^{n-1,2}} & K\uparrow & & \\
 & & \cdots & & \cdots & & \cdots & & \cdots & & \\
 & & K\uparrow & \nearrow^{S^{0,N}} & K\uparrow & \nearrow^{S^{1,N}} & K\uparrow & \nearrow^{S^{n-1,N}} & K\uparrow & & \\
0 & \longrightarrow & Z^{0,N} & \xrightarrow{d^{0,N}} & Z^{1,N} & \xrightarrow{d^{1,N}} & \cdots & \xrightarrow{d^{n-1,N}} & Z^{n,N} & \longrightarrow & 0
\end{array}
$$

Assumptions:

$$S^{i+1,j-1} \circ S^{i,j} = 0, \quad \forall i, j \geq 0,$$
$$S^{i,j} = d^{i,j-1}K^{i,j} - K^{i+1,j}d^{i,j} \quad S^{i,j-1}K^{i,j} = K^{i+1,j-1}S^{i,j}.$$

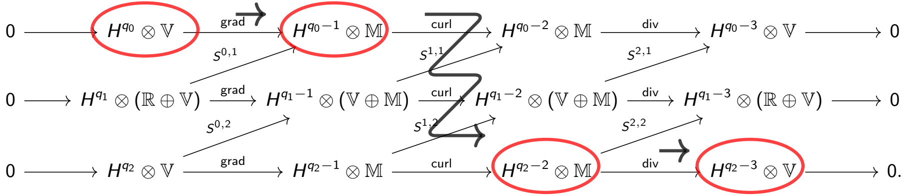
Corollary: $dS = -Sd$.

- $\rightarrow$: de-Rham complex, $\nearrow$: $(V^\bullet, S^\bullet)$, Lie algebra (co)homology
- algebraic Hodge decomposition

  $$Z^{i,j} = \mathcal{R}(S^{i-1,j+1}) \oplus \mathcal{R}(S^{i-1,j+1})^\perp = \mathcal{R}(S^{i-1,j+1}) \oplus \mathcal{N}(S^{i,j})^\perp \oplus H^{i,j},$$

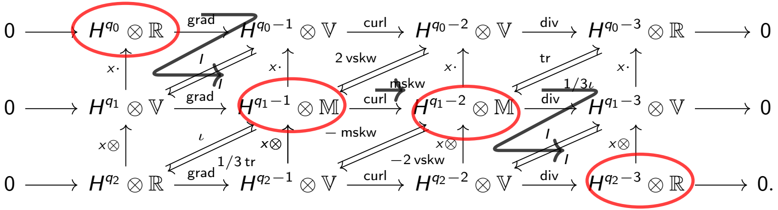
  where $H^{i,j} := \mathcal{R}(S^{i-1,j+1})^\perp \cap \mathcal{N}(S^{i,j})$ is an analogy of the space of harmonic forms. These "algebraic harmonic forms" will be the spaces in the BGG sequences.

conformal deformation complex:



$$0 \longrightarrow H^{q} \otimes \mathbb{V} \xrightarrow{\text{dev def}} H^{q-1} \otimes (\mathbb{S} \cap \mathbb{T}) \xrightarrow{\text{cot}} H^{q-4} \otimes (\mathbb{S} \cap \mathbb{T}) \xrightarrow{\text{div}} H^{q-5} \otimes \mathbb{V} \longrightarrow 0$$

conformal Hessian complex:



$$0 \longrightarrow H^{q} \otimes \mathbb{R} \xrightarrow{\text{dev hess}} H^{q-2} \otimes (\mathbb{S} \cap \mathbb{T}) \xrightarrow{\text{curl}} H^{q-3} \otimes (\mathbb{S} \cap \mathbb{T}) \xrightarrow{\text{div div}} H^{q-5} \otimes \mathbb{R} \longrightarrow 0$$

$\mathbb{M}$: matrix

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H^q \otimes \mathbb{V} & \xrightarrow{\text{grad}} & H^{q-1} \otimes \mathbb{M} & \xrightarrow{\text{rot}} & H^{q-2} \otimes \mathbb{V} & \longrightarrow & 0 \\
 & & & \overset{-\perp}{\nearrow} & & \overset{\text{tr}}{\nearrow} & & & \\
0 & \longrightarrow & H^{q-1} \otimes (\mathbb{R} \times \mathbb{R}) & \xrightarrow{\text{grad}} & H^{q-2} \otimes (\mathbb{V} \times \mathbb{V}) & \xrightarrow{\text{rot}} & H^{q-3} \otimes (\mathbb{R} \times \mathbb{R}) & \longrightarrow & 0 \\
 & & & \overset{-\iota}{\nearrow} & & \overset{I}{\nearrow} & & & \\
0 & \longrightarrow & H^{q-2} \otimes \mathbb{V} & \xrightarrow{\text{grad}} & H^{q-3} \otimes \mathbb{M} & \xrightarrow{\text{rot}} & H^{q-4} \otimes \mathbb{V} & \longrightarrow & 0.
\end{array}
$$

$$
0 \longrightarrow H^q \otimes \mathbb{V} \xrightarrow{D^0} \begin{pmatrix} H^{q-1} \otimes (\mathbb{S} \cap \mathbb{T}) \\ H^{q-3} \otimes (\mathbb{S} \cap \mathbb{T}) \end{pmatrix} \xrightarrow{D^1} H^{q-3} \longrightarrow 0,
$$

where, with pseudo-inverse $T^\bullet$,

$$
D^0 := \begin{pmatrix} \operatorname{dev}\operatorname{def} \\ \operatorname{grad} T^{1,1} \operatorname{grad} T^{1,0} \operatorname{grad} \end{pmatrix}, \quad D^1 = \begin{pmatrix} \operatorname{rot} T^{2,1} \operatorname{rot} T^{2,0} \operatorname{rot} \\ \operatorname{rot} \end{pmatrix}^T.
$$

Fixing 2D conformal Korn inequality:

$$
\|u\|_3 \le C(\|\operatorname{dev}\operatorname{def} u\|_2 + \|\operatorname{grad} T^{1,1} \operatorname{grad} T^{1,0} \operatorname{grad} u\|), \quad \forall u \perp \mathcal{N}.
$$

$$
\operatorname{dev}\operatorname{def} u = \begin{pmatrix} \frac{1}{2}(\partial_x u_1 - \partial_y u_2) & \frac{1}{2}(\partial_y u_1 + \partial_x u_2) \\ \frac{1}{2}(\partial_y u_1 + \partial_x u_2) & -\frac{1}{2}(\partial_x u_1 - \partial_y u_2) \end{pmatrix}
$$

Cauchy-Riemann operator. 2D conformal inequality with only C-R operator is invalid (Dain 2006). It can be fixed by including a third order term (Möbius structure).

Conclusion: cohomology of these complexes is isomorphic to the de Rham cohomology ✓

BGG is more than the complexes - it also provides machinery to derive results for elasticity, geometry... from results for de Rham.

Example: bounded Poincaré operators.

Poincaré operators: definition and motivation

Poincaré operators:

$$\cdots \rightleftarrows V^{i-1} \underset{P^i}{\overset{d^{i-1}}{\rightleftarrows}} V^i \underset{P^{i+1}}{\overset{d^i}{\rightleftarrows}} V^{i+1} \rightleftarrows \cdots$$

$P^k : V^k \mapsto V^{k-1}$, satisfying null-homotopy property

$$d^{k-1}P^k + P^{k+1}d^k = I_{V^k},$$

Motivation 1: constructing exact sequences and finite elements
(Hiptmair 1999, Arnold,Falk,Winther 2006)

$$du = 0 \implies u = (dP + Pd)u = d(Pu).$$

e.g., local exactness of de-Rham complexes.

Examples:

$$\cdots \longrightarrow \mathcal{P}_{r-(k-1)}\Lambda^{k-1} \xrightarrow{d^{k-1}} \mathcal{P}_{r-k}\Lambda^k \xrightarrow{d^k} \mathcal{P}_{r-(k+1)}\Lambda^{k+1} \longrightarrow \cdots$$

$$\cdots \longrightarrow \mathcal{P}_r\Lambda^{k-1} + P^k\mathcal{P}_r\Lambda^k \xrightarrow{d^{k-1}} \mathcal{P}_r\Lambda^k + P^{k+1}\mathcal{P}_r\Lambda^{k+1} \xrightarrow{d^k} \mathcal{P}_r\Lambda^k + P^{k+1}\mathcal{P}_r\Lambda^{k+1} \longrightarrow \cdots$$

Motivation 2: p-robustness of finite element methods

bounded, *polynomial-preserving* Poincaré operators imply results that are uniform with the polynomial degree.

Motivation 3: well-posedness of Stokes problem

given $f \in L^2$, find $u \in [H_0^1]^n$ and $p \in L^2/\mathbb{R}$, such that

$$\begin{aligned} -\Delta u + \nabla p &= f, \\ \operatorname{div} u &= 0. \end{aligned}$$

inf-sup condition: for any $q \in L^2/\mathbb{R}$, $\exists\ u = P(q) \in [H_0^1]^n$, s.t. $\operatorname{div} u = q$, $\|u\|_1 \leq C\|q\|$.

Motivation 4: analytic results, e.g., regular decomposition, compactness.

How to construct

Smooth de-Rham complex (see books on manifolds)

Let $F_t : \Omega \to \Omega, t \in [0,1]$ be a continuous family of operators indexed by t.
If u is a k-form:

$$(\mathfrak{p}[F]u)_x(\xi_2, \ldots, \xi_k) = \int_0^1 u_{F_t(x)}(\partial_t F_t(x), DF_t(x)\xi_2, \cdots, DF_t(x)\xi_k)\, dt.$$

Suppose that F_1 is identity and F_0 is constant x_0, then we have $d\mathfrak{p} + \mathfrak{p}d = I$ for $k \geq 1$ and $\mathfrak{p}du(x) = u(x) - u(x_0)$ for $k = 0$.

Simplification

- 1D with *base point* x_0

$$\mathfrak{p}(u) := \int_{x_0}^x u(y)\, dy, \quad \partial\mathfrak{p}(u) = u,\ \mathfrak{p}(\partial v) = v(x) - v(x_0).$$

- 3D vector proxy (choose a curve $\gamma(t) = tx$ connecting 0 and x):

$$\mathfrak{p}^1 u = \int_0^1 u_{tx} \cdot x dt, \quad \mathfrak{p}^2 v = \int_0^1 t v_{tx} \wedge x dt, \quad \mathfrak{p}^3 w = \int_0^1 t^2 w_{tx} x dt.$$

Sobolev de-Rham complex: Costabel-McIntosh 2010

$$0 \longrightarrow H^q\Lambda^0 \xrightarrow{d^0} H^{q-1}\Lambda^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} H^{q-n}\Lambda^n \longrightarrow 0,$$

where q is any real number.

- regularized Poincaré operators: averaging the base points in the smooth de-Rham version, mapping $H^q\Lambda^k$ to $H^{q+1}\Lambda^{k-1}$, polynomial-preserving
- generalized Bogolvskiĭ operators: "dual" of Poincaré, , mapping $H^q_0\Lambda^k$ to $H^{q+1}_0\Lambda^{k-1}$

pseudo-differential operators of order -1, which implies boundedness between various spaces (Sobolev, Besov...)

General Lipschitz domain:

$$dP + Pd = I - L,$$

where $L(u) \in C^\infty$ for any u. The smoothing operator L comes from partition of unity (into a union of star-shaped patches).

Goal: BGG version

Outline:

$$
\begin{array}{ccc}
Y^i & \underset{d^i}{\overset{P^{i+1}}{\rightleftarrows}} & Y^{i+1} \\
F \downarrow & & F \downarrow \\
Y^i & \underset{d^i_V}{\overset{P^{i+1}_V}{\rightleftarrows}} & Y^{i+1} \\
B^i \Downarrow \Uparrow A^i & & B^{i+1} \Downarrow \Uparrow A^{i+1} \\
\Upsilon^i & \underset{\mathscr{D}^i}{\overset{\mathscr{P}^{i+1}}{\rightleftarrows}} & \Upsilon^{i+1}
\end{array}
$$

$(Y^\bullet, d^\bullet)$: de Rham complex

$(Y^\bullet, d^\bullet_V)$: twisted complex

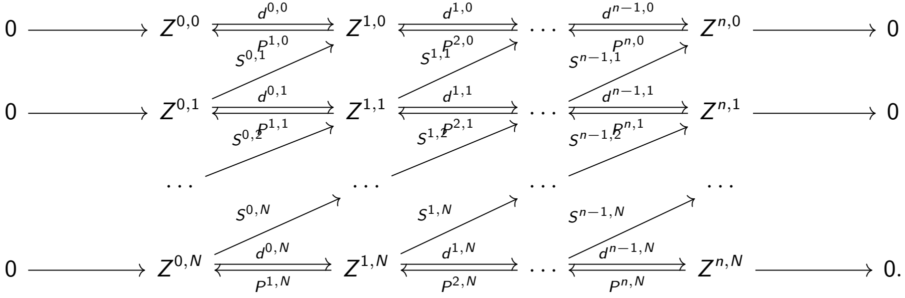
$(\Upsilon^\bullet, \mathscr{D}^\bullet)$: BGG complex

Poincaré operators for BGG complexes:

$$\mathscr{P} := B \circ F \circ P \circ F^{-1} \circ A.$$

If $P^\bullet$ satisfies $dP + Pd = I$, then $\mathscr{P}^\bullet$ satisfies $\mathscr{D}\mathscr{P} + \mathscr{P}\mathscr{D} = I$.

$F^i := (I - P^{i+1}S^i)^{-1} = \sum_{l=0}^{\infty}(P^{i+1}S^i)^l$



Twisted complexes

$$\cdots \longrightarrow Y^{i-1} \xrightarrow{d_V^{i-1}} Y^i \xrightarrow{d_V^i} Y^{i+1} \longrightarrow \cdots,$$

$$Y^i := \begin{pmatrix} Y^{i,0} \\ Y^{i,1} \\ \vdots \\ Y^{n-1,0} \\ Y^{n,0} \end{pmatrix}, \quad d_V := d - S = \begin{pmatrix} d^{i,0} & -S^{i,1} & & & \\ & d^{i,1} & -S^{i,2} & & \\ & & \ddots & \ddots & \\ & & & d^{i,n-1} & -S^{i,n} \\ & & & & d^{i,n} \end{pmatrix}$$

- twisted complexes: Timoshenko beam, Reissner-Mindlin plate, Cosserat elasticity
- BGG complexes: Euler-Bernoulli beam, Kirchhoff-Love plate, standard elasticity

1D example

BGG diagram

$$\begin{array}{ccc} H^q & \xrightarrow{\partial} & H^{q-1} \\ & \nearrow_{I} & \\ H^{q-1} & \xrightarrow{\partial} & H^{q-2}, \end{array}$$

where $\partial := \frac{d}{dx}$.

BGG complex:

$$H^q \xrightarrow{\partial^2} H^{q-2}$$

Poincaré operators: with $P_\sharp : H^{q-1} \to H^q$ and $P_\flat : H^{q-2} \to H^{q-1}$.

$$\mathscr{P} := P_\sharp P_\flat .$$

satisfying

$$\partial^2 \mathscr{P} = I_{H^{q-2}}, \quad \mathscr{P}\partial^2 = I_{H^q} + \alpha,$$

where $\alpha \in \mathcal{N}(\partial^2)$.

3D elasticity complex

$\mathscr{P}^1 = P(PS - SP)Td$ (all ingredients given in BGG diagram).

For *smooth e* (Christiansen,KH,Sande 2019):

$$\mathscr{P}^1(e) = \int_0^x e(y)\cdot\, dy - \underbrace{\int_0^x (\text{mskw} \int_0^y \underbrace{(e(z) \times \nabla)^T}_{Td(e)} \cdot dz)}_{SPTd(e)} \cdot dy.$$

For *general Sobolev functions*: generalization of Cesàro-Volterra formula (1906, 1907), satisfying

$$\mathscr{P}^1 \operatorname{def}(w) = w(x) - w(0) + \frac{1}{2}\int_0^x dy \wedge \nabla w(y).$$

Compared to

- '*A Cesàro-Volterra formula with little regularity*', Ciarlet, Gratie, Mardare, 2010 JMPA,

the new formulas are explicit, polynomial-preserving, work for a broad class of functions, for the entire complex.

Complex property $P \circ P = 0$?

Let $P^\bullet$ be Poincaré operators satisfying $dP + Pd = I$, but not necessarily $P \circ P = 0$. We can generally modify $P^\bullet$ to $\tilde{P}^\bullet$, defined by $\tilde{P} := P - DP^2 = PDP$. Then straightforward algebra implies $d\tilde{P} + \tilde{P}d = I$ and $\tilde{P} \circ \tilde{P} = 0$.

Nontrivial cohomology?

Standard procedure: cover $\Omega = \cup_j \Omega_j$, partition of unity, $1 = \sum_j \xi_j$ with ξ_j supported on Ω_j. Use $dP + Pd = I$ on each Ω_j, leading to $dP + Pd = I - L$ globally, where L is a smoothing operator.

Polynomial-preservation?

BGG Poincaré operators are polynomial-preserving if the de Rham versions are so (e.g., regularized Poincaré).

Some applications

Construction of polynomial exact sequences, e.g.,

$$\mathrm{RM} \longrightarrow \mathcal{P}_r \otimes \mathbb{V} \xrightarrow{\text{def}} \mathcal{P}_{r-1} \otimes \mathbb{S} \xrightarrow{\text{inc}} \mathcal{P}_{r-3} \otimes \mathbb{S} \xrightarrow{\text{div}} \mathcal{P}_{r-4} \otimes \mathbb{V} \longrightarrow 0.$$

One may also derive $\mathcal{P}_r^-$ type spaces (Raviart-Thomas etc.), which have been applied to constructing numerical schemes (Zhao et al. 2023), e.g.,

$$0 \longrightarrow \mathcal{P}_r \otimes \mathbb{V} + \mathscr{P}^1[\mathcal{P}_r \otimes \mathbb{S}] \xrightarrow{\text{def}} \mathcal{P}_r \otimes \mathbb{S} + \mathscr{P}^2[\mathcal{P}_r \otimes \mathbb{S}] \xrightarrow{\text{inc}} \mathcal{P}_r \otimes \mathbb{S} + \mathscr{P}^3[\mathcal{P}_r \otimes \mathbb{V}] \xrightarrow{\text{div}} \mathcal{P}_r \otimes \mathbb{V} \longrightarrow 0.$$

hp finite elements.

For any $v \in L^2 \otimes \mathbb{V}$, $\exists \sigma = \mathscr{P}(v) \in H^1(\mathbb{S})$, s.t., $\operatorname{div} \sigma = v$, and $\mathscr{P}$ is polynomial-preserving.

$\Longrightarrow$ *hp* stability for FEM for Hellinger-Reissner formulation (Aznaran, KH, Parker, in preparation)

References:

- *Complexes from complexes*, Douglas Arnold, KH; *Foundations of Computational Mathematics* (2021). complexes from two rows
- *BGG sequences with weak regularity and applications*, Andreas Čap, KH; *to appear, Foundations of Computational Mathematics* (2022) multi-rows
- *Poincaré path integrals for elasticity*, Snorre Christiansen, KH, Espen Sande, *Journal de Mathématiques Pures et Appliquées*, (2019) smooth elasticity complex in 3D
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