Tensor product finite element BGG complexes

Kaibo Hu
joint work with Francesca Bonizzoni (Milano), Guido Kanschat (Heidelberg), Duygu Sap (Oxford)

University of Oxford

Biennial Numerical Analysis Conference

June 2023, University of Strathclyde

de Rham complexes

$$0 \longrightarrow C^\infty(\Omega) \xrightarrow{\text{grad}} C^\infty(\Omega;\mathbb{R}^3) \xrightarrow{\text{curl}} C^\infty(\Omega;\mathbb{R}^3) \xrightarrow{\text{div}} C^\infty(\Omega) \longrightarrow 0.$$

$$\cdots \longrightarrow C^\infty\Lambda^{k-1} \xrightarrow{d^{k-1}} C^\infty\Lambda^k \xrightarrow{d^k} C^\infty\Lambda^{k+1} \longrightarrow \cdots$$

elasticity complex

displacement formulation

Kröner's continuum description of dislocations/defects, internal stress

$$0 \longrightarrow C^\infty \otimes \mathbb{V} \xrightarrow{\text{def}} C^\infty \otimes \mathbb{S} \xrightarrow{\text{inc}} C^\infty \otimes \mathbb{S} \xrightarrow{\text{div}} C^\infty \otimes \mathbb{V} \longrightarrow 0$$

displacement *strain (metric)* *stress (curvature)* *force*

intrinsic elasticity (Ciarlet et al.)

Hellinger-Reissner principle of elasticity

$\mathbb{V} := \mathbb{R}^3$ vectors, $\mathbb{S} := \mathbb{R}^{3\times 3}_{\text{sym}}$ symmetric matrices

$$\text{def}\, u := 1/2(\nabla u + \nabla u^T), \quad \text{inc}\, g := \nabla \times g \times \nabla, \quad \text{div}\, v := \nabla \cdot v.$$

g metric $\Rightarrow$ $\text{inc}\, g$ linearized Einstein tensor ($\backsimeq$ Riem $\backsimeq$ Ric in 3D)

$\text{inc} \circ \text{def} = 0$: Saint-Venant compatibility

$\text{div} \circ \text{inc} = 0$: Bianchi identity

Bernstein-Gelfand-Gelfand (BGG) construction:

B-G-G 1975, Eastwood 1999, Čap, Slovák, Souček 2001, Arnold, Falk, Winther 2006, Arnold, Hu 2021, Čap, Hu 2022.

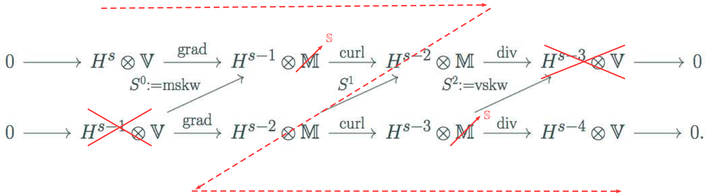
Continuous level

$$0 \longrightarrow H^2 \xrightarrow{\partial_x^2} L^2 \longrightarrow 0.$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^2 & \xrightarrow{\partial_x} & H^1 & \longrightarrow & 0 \\ & & & \overset{I}{\nearrow} & & & \\ 0 & \longrightarrow & H^1 & \xrightarrow{\partial_x} & L^2 & \longrightarrow & 0. \end{array}$$

- two de-Rham complexes with different continuity,
- cohomology: $\mathcal{N}(\partial_x^2) \cong \mathcal{N}(\partial_x) \oplus \mathcal{N}(\partial_x)$, ∂_x^2 is onto.

Algebraic and analytic construction (Arnold, KH 2021): derive elasticity from deRham



$S^1 u := u^T - \mathrm{tr}(u) I.$

output: elasticity complex

$$0 \longrightarrow H^s \otimes \mathbb{V} \xrightarrow{\text{def}} H^{s-1} \otimes \mathbb{S} \xrightarrow{\text{curl}\,\mathrm{T}\,\text{curl}} H^{s-3} \otimes \mathbb{S} \xrightarrow{\text{div}} H^{s-4} \otimes \mathbb{V} \longrightarrow 0.$$

de Rham results + homological algebra $\Rightarrow$ elasticity/geometry results

Consequence: cohomology of derived complex is isomorphic to de Rham (which implies analytic properties)

Example: de Rham complexes with "double indices"

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & H^q \otimes \mathrm{Alt}^{0,0} & \xrightarrow{d} & H^{q-1} \otimes \mathrm{Alt}^{1,0} & \xrightarrow{d} & \cdots & \xrightarrow{d} & H^{q-n} \otimes \mathrm{Alt}^{n,0} & \longrightarrow & 0 \\
 & & & \nearrow^{s^{0,1}} & & \nearrow^{s^{1,1}} & & \nearrow^{s^{n-1,1}} & & & \\
0 & \longrightarrow & H^{q-1} \otimes \mathrm{Alt}^{0,1} & \xrightarrow{d} & H^{q-2} \otimes \mathrm{Alt}^{1,1} & \xrightarrow{d} & \cdots & \xrightarrow{d} & H^{q-n-1} \otimes \mathrm{Alt}^{n,1} & \longrightarrow & 0 \\
 & & \vdots & & \vdots & & & & \vdots & & \\
0 & \longrightarrow & H^{q-n+1} \otimes \mathrm{Alt}^{0,n-1} & \xrightarrow{d} & H^{q-n} \otimes \mathrm{Alt}^{1,n-1} & \xrightarrow{d} & \cdots & \xrightarrow{d} & H^{q-2n+1} \otimes \mathrm{Alt}^{n,n-1} & \longrightarrow & 0 \\
 & & & \nearrow^{s^{0,n}} & & \nearrow^{s^{1,n}} & & \nearrow^{s^{n-1,n}} & & & \\
0 & \longrightarrow & H^{q-n} \otimes \mathrm{Alt}^{0,n} & \xrightarrow{d} & H^{q-n-1} \otimes \mathrm{Alt}^{1,n} & \xrightarrow{d} & \cdots & \xrightarrow{d} & H^{q-2n} \otimes \mathrm{Alt}^{n,n} & \longrightarrow & 0.
\end{array}
$$

where $\mathrm{Alt}^{i,J} := \mathrm{Alt}^{i} \otimes \mathrm{Alt}^{J}$. $\quad S^{i,j} : \mathrm{Alt}^{i,j} \to \mathrm{Alt}^{i+1,j-1}$

$$
s^{i,J}\mu(v_0, \cdots, v_i)(w_1, \cdots, w_{J-1}) := \sum_{l=0}^{i} (-1)^l \mu(v_0, \cdots, \widehat{v_l}, \cdots, v_i)(v^l, w_1, \cdots, w_{J-1}),
$$

$$
\forall v_0, \cdots, v_i, w_1, \cdots, w_{J-1} \in \mathbb{R}^n.
$$

More 3D examples:
(diagonal maps: bijective; superdiagonal: surjective; subdiagonal: injective.)

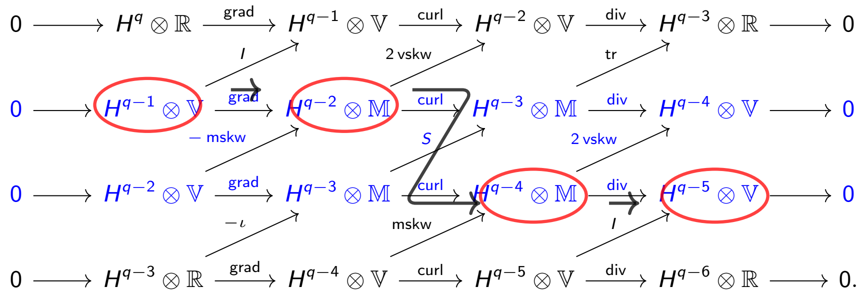
$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & H^q \otimes \mathbb{R} & \xrightarrow{\text{grad}} & H^{q-1} \otimes \mathbb{V} & \xrightarrow{\text{curl}} & H^{q-2} \otimes \mathbb{V} & \xrightarrow{\text{div}} & H^{q-3} \otimes \mathbb{R} & \longrightarrow & 0 \\
 & & & \nearrow^{I} & & \nearrow^{2\,\text{vskw}} & & \nearrow^{\text{tr}} & & & \\
0 & \longrightarrow & H^{q-1} \otimes \mathbb{V} & \xrightarrow{\text{grad}} & H^{q-2} \otimes \mathbb{M} & \xrightarrow{\text{curl}} & H^{q-3} \otimes \mathbb{M} & \xrightarrow{\text{div}} & H^{q-4} \otimes \mathbb{V} & \longrightarrow & 0 \\
 & & & \nearrow_{-\,\text{mskw}} & & \nearrow_{S} & & \nearrow_{2\,\text{vskw}} & & & \\
0 & \longrightarrow & H^{q-2} \otimes \mathbb{V} & \xrightarrow{\text{grad}} & H^{q-3} \otimes \mathbb{M} & \xrightarrow{\text{curl}} & H^{q-4} \otimes \mathbb{M} & \xrightarrow{\text{div}} & H^{q-5} \otimes \mathbb{V} & \longrightarrow & 0 \\
 & & & \nearrow^{-\iota} & & \nearrow^{\text{mskw}} & & \nearrow^{I} & & & \\
0 & \longrightarrow & H^{q-3} \otimes \mathbb{R} & \xrightarrow{\text{grad}} & H^{q-4} \otimes \mathbb{V} & \xrightarrow{\text{curl}} & H^{q-5} \otimes \mathbb{V} & \xrightarrow{\text{div}} & H^{q-6} \otimes \mathbb{R} & \longrightarrow & 0.
\end{array}
$$

Hessian complex:

$$
0 \longrightarrow H^q \otimes \mathbb{R} \xrightarrow{\text{hess}} H^{q-2} \otimes \mathbb{S} \xrightarrow{\text{curl}} H^{q-3} \otimes \mathbb{T} \xrightarrow{\text{div}} H^{q-4} \otimes \mathbb{V} \longrightarrow 0.
$$

biharmonic equations, plate theory, Einstein-Bianchi system of general relativity

More 3D examples:
(diagonal maps: bijective; superdiagonal: surjective; subdiagonal: injective.)



elasticity complex:

$$0 \longrightarrow H^{q-1} \otimes \mathbb{V} \xrightarrow{\text{def}} H^{q-2} \otimes \mathbb{S} \xrightarrow{\text{inc}} H^{q-4} \otimes \mathbb{S} \xrightarrow{\text{div}} H^{q-5} \otimes \mathbb{V} \longrightarrow 0.$$

elasticity, defects, metric, curvature

More 3D examples:
(diagonal maps: bijective; superdiagonal: surjective; subdiagonal: injective.)

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & H^q \otimes \mathbb{R} & \xrightarrow{\text{grad}} & H^{q-1} \otimes \mathbb{V} & \xrightarrow{\text{curl}} & H^{q-2} \otimes \mathbb{V} & \xrightarrow{\text{div}} & H^{q-3} \otimes \mathbb{R} & \longrightarrow & 0 \\
 & & & \overset{I}{\nearrow} & & \overset{2\,\text{vskw}}{\nearrow} & & \overset{\text{tr}}{\nearrow} & & & \\
0 & \longrightarrow & H^{q-1} \otimes \mathbb{V} & \xrightarrow{\text{grad}} & H^{q-2} \otimes \mathbb{M} & \xrightarrow{\text{curl}} & H^{q-3} \otimes \mathbb{M} & \xrightarrow{\text{div}} & H^{q-4} \otimes \mathbb{V} & \longrightarrow & 0 \\
 & & & \overset{-\text{mskw}}{\nearrow} & & \overset{S}{\nearrow} & & \overset{2\,\text{vskw}}{\nearrow} & & & \\
0 & \longrightarrow & H^{q-2} \otimes \mathbb{V} & \xrightarrow{\text{grad}} & H^{q-3} \otimes \mathbb{M} & \xrightarrow{\text{curl}} & H^{q-4} \otimes \mathbb{M} & \xrightarrow{\text{div}} & H^{q-5} \otimes \mathbb{V} & \longrightarrow & 0 \\
 & & & \overset{-\iota}{\nearrow} & & \overset{\text{mskw}}{\nearrow} & & \overset{I}{\nearrow} & & & \\
0 & \longrightarrow & H^{q-3} \otimes \mathbb{R} & \xrightarrow{\text{grad}} & H^{q-4} \otimes \mathbb{V} & \xrightarrow{\text{curl}} & H^{q-5} \otimes \mathbb{V} & \xrightarrow{\text{div}} & H^{q-6} \otimes \mathbb{R} & \longrightarrow & 0.
\end{array}
$$

divdiv complex:

$$0 \longrightarrow H^{q-2} \otimes \mathbb{V} \xrightarrow{\text{dev grad}} H^{q-3} \otimes \mathbb{T} \xrightarrow{\text{sym curl}} H^{q-4} \otimes \mathbb{S} \xrightarrow{\text{div div}} H^{q-6} \otimes \mathbb{R} \longrightarrow 0.$$

plate theory, elasticity

Discretization of complexes:

- 2D stress: Arnold-Winther 2002, J.Hu-S.Zhang 2014, Christiansen-KH 2018, Chen-Huang 2022 (arbitrary regularity)
- 2D strain: Chen-J.Hu-Huang 2014 (Regge/HHJ), Christiansen-KH 2018 (conforming), Chen-Huang 2020, DiPietro-Droniou 2021 (polygonal meshes)
- 3D elasticity: various results on last part of complex, Hauret-Kuhl-Ortiz 2007 (discrete geometry/mechanics), Arnold-Awanou-Winther 2008, Christiansen 2011 (Regge), Christiansen-Gopalakrishnan-Guzmán-Hu 2020, Chen-Huang 2021, J.Hu-Y.Liang-T.Lin 2023 (cubical)
- 3D Hessian: Chen-Huang 2020, J.Hu-Liang 2021, Arf-Simeon 2021 (splines), J.Hu-Y.Liang-T.Lin 2023 (cubical)
- 3D divdiv: Chen-Huang 2021, J.Hu-Liang-Ma 2021, Sander 2021 ($H(\operatorname{sym}\operatorname{curl})$, $H(\operatorname{dev}\operatorname{sym}\operatorname{curl})$), J.Hu-Liang-Ma-Zhang 2022, DiPietro-Hanot 2023 (polyhedral mesh)
- nD: Chen-Huang 2021 (last two spaces)
- conformal complexes: open.

Question: discrete BGG complexes in any dimension for any degree

Idea of tensor product construction: approximate $f(x,y)$ by $u(x)v(y)$.
Question: how does this separation of variables interact with other structures (Sobolev spaces, approximation, *sequences and cohomology, interpolants*...)

de Rham complexes: Arnold,Boffi,Bonizzoni 2013

$$\mathcal{Q}_r^- \Lambda^k(I^n) = \bigoplus_{\sigma \in \Sigma(k;n)} \left[\bigotimes_{i=1}^{n} \mathcal{P}_{r-\delta_{i,\sigma}}(I) \right] dx^{\sigma_1} \wedge \cdots \wedge dx^{\sigma_k},$$

where

$$\delta_{i,\sigma} = \begin{cases} 1, & i \in \{\sigma_1, \cdots, \sigma_k\}, \\ 0, & \text{otherwise}. \end{cases}$$

higher form degree corresponds to lower polynomial degree (in a delicate way).
One verifies that the above discrete spaces are compatible with exterior derivatives:

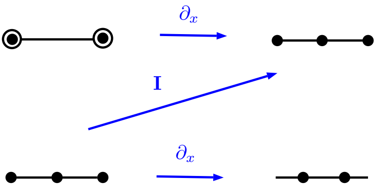
$$d^k \mathcal{Q}_r^- \Lambda^k(I^n) \subset \mathcal{Q}_r^- \Lambda^{k+1}(I^n).$$

Example: $\mathcal{P}_{r,s} := \mathcal{P}_r \otimes \mathcal{P}_s$

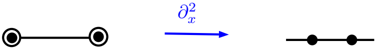
$$0 \longrightarrow \mathcal{P}_{r,r} \xrightarrow{\text{grad}} \begin{pmatrix} \mathcal{P}_{r-1,r} \\ \mathcal{P}_{r,r-1} \end{pmatrix} \xrightarrow{\text{rot}} \mathcal{P}_{r-1,r-1} \longrightarrow 0$$

Variations/extensions: global spline (Buffa,Rivas,Sangalli,Vázquez 2011, 3D) or finite element spaces and interpolants (Bonizzoni,Kanschat 2021); FES (Christiansen 2009), *possible to start with anisotropic space* $\mathcal{P}_{r_1,r_2}$

BGG complexes



implies



Input: two 1D diagrams, $i = 1, 2$, $\mathcal{S}_r^q$: spline space with regularity index r, polynomial degree q.

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{S}_{r_i}^{p_i}(\mathcal{I}) & \xrightarrow{\partial} & \mathcal{S}_{r_i-1}^{p_i-1}(\mathcal{I}) & \longrightarrow & 0 \\
 & & & \overset{I}{\nearrow} & & & \\
0 & \longrightarrow & \mathcal{S}_{r_i-1}^{p_i-1}(\mathcal{I}) & \xrightarrow{\partial} & \mathcal{S}_{r_i-2}^{p_i-2}(\mathcal{I}) & \longrightarrow & 0,
\end{array}
$$

Examples in 2D: $\mathcal{S}_{r_1,r_2}^{p_1,p_2} := \mathcal{S}_{r_1}^{p_1} \otimes \mathcal{S}_{r_2}^{p_2}$

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{S}_{r_1,r_2}^{p_1,p_2} & \xrightarrow{\text{grad}} & \begin{pmatrix} \mathcal{S}_{r_1-1,r_2}^{p_1-1,p_2} \\ \mathcal{S}_{r_1,r_2-1}^{p_1,p_2-1} \end{pmatrix} & \xrightarrow{\text{rot}} & \mathcal{S}_{r_1-1,r_2-1}^{p_1-1,p_2-1} & \longrightarrow & 0 \\
 & & & \overset{\mathrm{I}}{\nearrow} & & \overset{\text{sskw}}{\nearrow} & & & \\
0 & \longrightarrow & \begin{pmatrix} \mathcal{S}_{r_1-1,r_2}^{p_1-1,p_2} \\ \mathcal{S}_{r_1,r_2-1}^{p_1,p_2-1} \end{pmatrix} & \xrightarrow{\text{grad}} & \begin{pmatrix} \mathcal{S}_{r_1-2,r_2}^{p_1-2,p_2} & \mathcal{S}_{r_1-1,r_2-1}^{p_1-1,p_2-1} \\ \mathcal{S}_{r_1-1,r_2-1}^{p_1-1,p_2-1} & \mathcal{S}_{r_1,r_2-2}^{p_1,p_2-2} \end{pmatrix} & \xrightarrow{\text{rot}} & \begin{pmatrix} \mathcal{S}_{r_1-2,r_2-1}^{p_1-2,p_2-1} \\ \mathcal{S}_{r_1-1,r_2-2}^{p_1-1,p_2-2} \end{pmatrix} & \longrightarrow & 0 \\
 & & & \overset{-\mathrm{mskw}}{\nearrow} & & \overset{S}{\nearrow} & & & \\
0 & \longrightarrow & \mathcal{S}_{r_1-1,r_2-1}^{p_1-1,p_2-1} & \xrightarrow{\text{grad}} & \begin{pmatrix} \mathcal{S}_{r_1-2,r_2-1}^{p_1-2,p_2-1} \\ \mathcal{S}_{r_1-1,r_2-2}^{p_1-1,p_2-2} \end{pmatrix} & \xrightarrow{\text{rot}} & \mathcal{S}_{r_1-2,r_2-2}^{p_1-2,p_2-2} & \longrightarrow & 0.
\end{array}
$$

- first row, first column: de Rham complexes,
- regularity decreases in each row and in each column, (larger form degrees correspond to lower regularity and polynomial degree)
- then S operators match spaces well.

Formalise the idea with forms: any dimension and any degree

$$\mathscr{S}_{\boldsymbol{r}}^{\boldsymbol{p}}\Lambda^{I,J} := \oplus_{(i_i,\cdots,i_n)\in\chi_I,(j_1,\cdots,j_n)\in\chi_J}(\mathcal{S}_{r_1-i_1-j_1}^{p_1-i_1-j_1}\Lambda^{i_1,j_1}\otimes\cdots\otimes\mathcal{S}_{r_n-i_n-j_n}^{p_n-i_n-j_n}\Lambda^{i_n,j_n}),$$

where sum for $i_1+\cdots+i_n = I$, $j_1+\cdots+j_n = J$; $\boldsymbol{p} = (p_1, p_2, \cdots, p_n)$,

$$\mathcal{S}_{r_l-i_l-j_l}^{p_l-i_l-j_l}\Lambda^{i,j}(\mathcal{I}) := \mathcal{S}_{r_l-i_l-j_l}^{p_l-i_l-j_l}\mathrm{Alt}^i\otimes\mathrm{Alt}^j.$$

The spaces are compatible with $d^\bullet$ and $S^\bullet$:

$$d^I\mathscr{S}_{\boldsymbol{r}}^{\boldsymbol{p}}\Lambda^{I,J}\subset\mathscr{S}_{\boldsymbol{r}}^{\boldsymbol{p}}\Lambda^{I+1,J},\quad S^{I,J}\mathscr{S}_{\boldsymbol{r}}^{\boldsymbol{p}}\Lambda^{I,J}\subset\mathscr{S}_{\boldsymbol{r}}^{\boldsymbol{p}}\Lambda^{I+1,J-1}.$$

Then we can run the BGG machinery with the diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \mathscr{S}_{\boldsymbol{r}}^{\boldsymbol{p}}\Lambda^{I-1,J-1} & \xrightarrow{d} & \mathscr{S}_{\boldsymbol{r}}^{\boldsymbol{p}}\Lambda^{I,J-1} & \xrightarrow{d} & \mathscr{S}_{\boldsymbol{r}}^{\boldsymbol{p}}\Lambda^{I+1,J-1} & \longrightarrow & \cdots \\
 & & & {\scriptstyle S^{I-1,J}}\nearrow & & {\scriptstyle S^{I,J}}\nearrow & & & \\
\cdots & \longrightarrow & \mathscr{S}_{\boldsymbol{r}}^{\boldsymbol{p}}\Lambda^{I-1,J} & \xrightarrow{d} & \mathscr{S}_{\boldsymbol{r}}^{\boldsymbol{p}}\Lambda^{I,J} & \xrightarrow{d} & \mathscr{S}_{\boldsymbol{r}}^{\boldsymbol{p}}\Lambda^{I+1,J} & \longrightarrow & \cdots .
\end{array}$$

Derivation of BGG complexes: same as continuous level, consisting of kernels and cokernels of S.

Tensor product finite elements: splines with local degrees of freedom

The construction of degrees of freedom and bounded commuting interpolations also comes from tensor products.

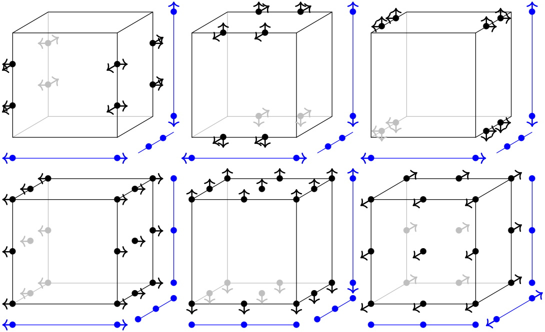


Figure: Degrees of freedom for the lowest order strain element $V^{1,1}$ of the elasticity complex. Diagonal entries $\sigma_{11}, \sigma_{22}, \sigma_{33}$ in the top row. Off-diagonal entries $\sigma_{23} = \sigma_{32}$, $\sigma_{13} = \sigma_{31}$, and $\sigma_{12} = \sigma_{21}$ in the bottom row. Pairs of arrows indicate first order and mixed second order derivatives.

Special cases (div div complex) coincide with J.Hu,Y.Liang,R.Ma,M.Zhang 2022.

Limitations

Less flexible for complicated geometry: even more challenging for BGG complexes. Straightforward pullbacks do not commute with differentials except for affine maps (while for de Rham, Piola maps commute with d's). Deeper reasons for this (BGG sequences are geometric, more than topological).

Major challenge for isogeometric analysis (Arf,Simeon 2021). Arf talk: ways to avoid this issue in IGA. Modified pullbacks can be obtained using the BGG machinery (KH,Sande,Toshniwal, in preparation).

Not clear for more complicated structures: $\mathbb{S} \cap \mathbb{T}$ symmetric & tracefree tensors

ker of dev def: conformal Killing v.f.

Cotton-York: flatness in conformal geometry

$$0 \longrightarrow H^s(\Omega) \otimes \mathbb{V} \xrightarrow{\text{dev def}} H^{s-1}(\Omega) \otimes (\mathbb{S} \cap \mathbb{T}) \xrightarrow{\text{Cott}} H^{s-4}(\Omega) \otimes (\mathbb{S} \cap \mathbb{T}) \xrightarrow{\text{div}} H^{s-5}(\Omega) \otimes \mathbb{V} \longrightarrow 0$$

gravitational wave variable: transverse-traceless (TT) gauge (= symmetric, trace-free, div-free)

stress like variable def$u$ in NS (Gopalakrishnan, Lederer, Schöberl, 2019)

It seems that we have used up all the flexibility in the indices of splines/finite elements and there is no obvious way to obtain both symmetry and tracelessness.

References:

- *Complexes from complexes*, Douglas Arnold, KH; *Foundations of Computational Mathematics* (2021). framework, analytic results from homological algebraic structures
- *BGG sequences with weak regularity and applications*, Andreas Čap, KH; *accepted, Foundations of Computational Mathematics* (2023)
  more general framework, conformal complexes, applications
- *Discrete tensor product BGG sequences: splines and finite elements*, Francesca Bonizzoni, KH, Guido Kanschat, Duygu Sap; *arxiv* (2023).
  any dimension, any degree