Finite element differential complexes

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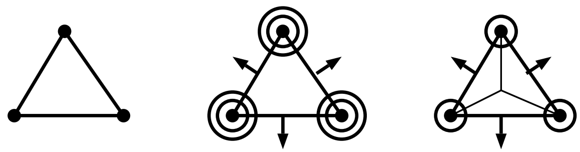
Homological Perspective on Splines and Finite Elements, BIRS workshop

Classical definition of finite elements:

splines (piecewise polynomials with certain continuity) that can be obtained by gluing local pieces.



C^0 C^1 C^1

- degrees of freedom (DoFs): minimal determining set (defined on one simplex)
- unisolvence

dot: function value, **circles**: derivative value, **arrows**: normal (tangential component) or normal derivative, depending on the context.

This talk: vector- and tensor-valued problems (Maxwell, Navier-Stokes, elasticity, Einstein...)

- The question can be asked for general splines, although most examples are (simplicial) finite elements.
- Focusing on general ideas in the literature, rather than individual results.
  Some references up to 2022 can be found in

Hu, K. (2022). Oberwolfach report: Discretization of Hilbert complexes. arXiv:2208.03420.

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Basic homological algebra

$$\cdots \longrightarrow V^{i-1} \xrightarrow{d^{i-1}} V^i \xrightarrow{d^i} V^{i+1} \longrightarrow \cdots$$

V^i: vector spaces, d^i: linear (or nonlinear) operators

- complex: $d^i V^i \subset V^{i+1}$, $d^{i+1} \circ d^i = 0$, $\forall i$, (implies $\mathcal{R}(d^{i-1}) \subset \mathcal{N}(d^i)$)
- exact: $\mathcal{N}(d^i) = \mathcal{R}(d^{i-1})$,
- cohomology (when d is linear): $\mathscr{H}^i := \mathcal{N}(d^i)/\mathcal{R}(d^{i-1})$.

Example: de Rham complex

$$0 \longrightarrow C^\infty\Lambda^0 \xrightarrow{d^0} C^\infty\Lambda^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} C^\infty\Lambda^n \longrightarrow 0.$$

$C^\infty\Lambda^k$: (smooth) k-forms, d^k: exterior derivative.

Vector proxies in 3D:

$$0 \longrightarrow C^\infty \xrightarrow{\mathrm{grad}} C^\infty \otimes \mathbb{R}^3 \xrightarrow{\mathrm{curl}} C^\infty \otimes \mathbb{R}^3 \xrightarrow{\mathrm{div}} C^\infty \longrightarrow 0.$$

Formal adjoint of operators:

$$\operatorname{grad}^* = -\operatorname{div}, \quad \operatorname{curl}^* = \operatorname{curl}, \quad \operatorname{div}^* = -\operatorname{grad}.$$

$\int_\Omega \operatorname{grad} u \cdot v = -\int_\Omega u \operatorname{div} v + \text{bound. term}, \quad \int_\Omega \operatorname{curl} u \cdot v = \int_\Omega u \cdot \operatorname{curl} v + \text{bound. term}$

$$(\operatorname{grad} u, v) = (u, -\operatorname{div} v), \qquad (\operatorname{curl} u, v) = (u, \operatorname{curl} v)$$

Formal adjoint of de Rham complex:

$$0 \longleftarrow C^\infty(\Omega) \xleftarrow{-\operatorname{div}} C^\infty(\Omega;\mathbb{R}^3) \xleftarrow{\operatorname{curl}} C^\infty(\Omega;\mathbb{R}^3) \xleftarrow{-\operatorname{grad}} C^\infty(\Omega) \longleftarrow 0.$$

$$d_2^* := -\operatorname{div}, \quad d_1^* := \operatorname{curl}, \quad d_0^* := -\operatorname{grad}.$$

Connections to PDEs: Hodge-Laplacian problems

$$(d^{k-1}d^*_{k-1} + d^*_k d^k)u = f.$$

Connections to PDEs: Hodge-Laplacian problems

$$(d^{k-1}d^*_{k-1} + d^*_k d^k)u = f.$$

$$0 \rightleftarrows C^\infty(\Omega) \underset{-\operatorname{div}}{\overset{\operatorname{grad}}{\rightleftarrows}} C^\infty(\Omega;\mathbb{R}^3) \qquad C^\infty(\Omega;\mathbb{R}^3) \qquad C^\infty(\Omega) \qquad 0.$$

Hodge-Laplacian problem:

$$-\operatorname{div}\operatorname{grad} u = f.$$

Poisson equation.

Connections to PDEs: Hodge-Laplacian problems

$$(d^{k-1}d^*_{k-1} + d^*_k d^k)u = f.$$

$$0 \qquad C^\infty(\Omega) \underset{-\operatorname{div}}{\overset{\operatorname{grad}}{\rightleftarrows}} C^\infty(\Omega;\mathbb{R}^3) \underset{\operatorname{curl}}{\overset{\operatorname{curl}}{\rightleftarrows}} C^\infty(\Omega;\mathbb{R}^3) \qquad C^\infty(\Omega) \qquad 0.$$

Hodge-Laplacian problem:

$$-\operatorname{grad}\operatorname{div} v + \operatorname{curl}\operatorname{curl} v = f.$$

Maxwell equations.

$$(d^{k-1}d^*_{k-1} + d^*_k d^k)u = f.$$

$$0 \qquad C^\infty(\Omega) \qquad C^\infty(\Omega;\mathbb{R}^3) \underset{\text{curl}}{\overset{\text{curl}}{\rightleftarrows}} C^\infty(\Omega;\mathbb{R}^3) \underset{-\,\text{grad}}{\overset{\text{div}}{\rightleftarrows}} C^\infty(\Omega) \qquad 0.$$

Hodge-Laplacian problem:

$$\operatorname{curl}\operatorname{curl} v - \operatorname{grad}\operatorname{div} v = f.$$

Maxwell equations.

Connections to PDEs: Hodge-Laplacian problems

$$(d^{k-1} d_{k-1}^* + d_k^* d^k) u = f.$$

$$0 \qquad C^\infty(\Omega) \qquad C^\infty(\Omega; \mathbb{R}^3) \qquad C^\infty(\Omega; \mathbb{R}^3) \underset{-\operatorname{grad}}{\overset{\operatorname{div}}{\rightleftarrows}} C^\infty(\Omega) \rightleftarrows 0.$$

Hodge-Laplacian problem:

$$-\operatorname{div} \operatorname{grad} u = f.$$

Poisson equation.

Use finite elements/splines in a complex!

$$
\begin{aligned}
(\operatorname{curl}\operatorname{curl}-\operatorname{grad}\operatorname{div})\boldsymbol{u} = -\Delta \boldsymbol{u} = \boldsymbol{f}, &\quad \text{in } \Omega,\\
u\cdot n = 0, &\quad \text{on } \partial\Omega,\\
\operatorname{curl} u \times n = 0, &\quad \text{on } \partial\Omega.
\end{aligned}
$$

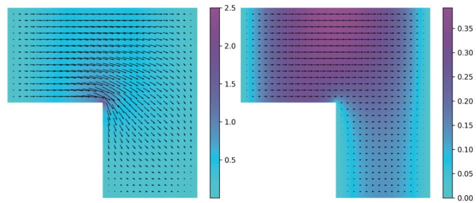
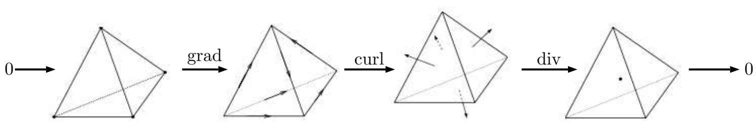


Figure 5.1. *Finite element solution to the Hodge Laplacian problem on an L-shaped domain (with $f = (1,0)$). The left figure is calculated with a mixed method which is known to converge to the solution in L^2. The right figure is based on the primal formulation using 24,576 piecewise linear elements. The primal-based numerical solution entirely misses the dominant behavior at the reentrant corner and produces a wholly inaccurate solution.*

Arnold, *Finite element exterior calculus*, SIAM, Chapter 5



Philosophy: vector/tensor problems cannot be treated as a collection of scalar problems for components.
Structures should be incorporated (differential structures, cohomology).

Many more examples. Using finite elements in a complex is important for

- avoiding spurious solutions of eigenvalue problems,
- parameter-robustness of iterative solvers,
- structure-preserving properties (divergence-free constraints etc.)
- ...

Things work when we discrete the entire complex and preserve the cohomology.
Goal of the rest of the talk: construct finite element (spline) complexes with the correct cohomology

Discrete Differential Forms (Bossavit 1988, Hiptmair 1999, ...),
Finite Element Exterior Calculus ("FEEC", Arnold, Falk, Winther 2006, ...)

- Arnold, D. N. (2018). Finite element exterior calculus. SIAM.
- Arnold, D. N., Falk, R. S., & Winther, R. (2006). Finite element exterior calculus, homological techniques, and applications. Acta numerica, 15, 1-155.
- Hiptmair, R. (2002). Finite elements in computational electromagnetism. Acta Numerica, 11, 237-339.

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Various operators and tensors

de Rham complex

$$0 \longrightarrow C^\infty\Lambda^0 \xrightarrow{d^0} C^\infty\Lambda^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} C^\infty\Lambda^n \longrightarrow 0,$$

with 3D vector proxies

$$0 \longrightarrow C^\infty \xrightarrow{\text{grad}} C^\infty \otimes \mathbb{R}^3 \xrightarrow{\text{curl}} C^\infty \otimes \mathbb{R}^3 \xrightarrow{\text{div}} C^\infty \longrightarrow 0.$$

Maxwell, Navier-Stokes

BERNSTEIN-GELFAND-GELFAND (BGG) COMPLEXES

$\mathbb{V} = \mathbb{R}^3$: vectors $\quad \mathbb{S} = \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ symmetric matrices $\quad \mathbb{T} = \mathbb{R}^{3\times 3}_{\mathrm{traceless}}$ traceless matrices

- Hessian complex:

$$0 \longrightarrow C^\infty \xrightarrow{\text{hess}} C^\infty \otimes \mathbb{S} \xrightarrow{\text{curl}} C^\infty \otimes \mathbb{T} \xrightarrow{\text{div}} C^\infty \otimes \mathbb{V} \longrightarrow 0.$$

  $\mathrm{hess} = \mathrm{grad}\,\mathrm{grad}$

  plate problem ($(\mathrm{hess}^*)\,\mathrm{hess} = \mathrm{div}\,\mathrm{div}\,\mathrm{hess} = \Delta^2$, biharmonic), Einstein-Bianchi formulation,

- elasticity (Calabi, Kröner, Riemannian deformation) complex:

$$0 \longrightarrow C^\infty \otimes \mathbb{V} \xrightarrow{\text{def}} C^\infty \otimes \mathbb{S} \xrightarrow{\text{inc}} C^\infty \otimes \mathbb{S} \xrightarrow{\text{div}} C^\infty \otimes \mathbb{V} \longrightarrow 0.$$

  $\mathrm{def} := \mathrm{sym}\,\mathrm{grad}$, $\mathrm{inc} := \mathrm{curl} \circ \mathrm{T} \circ \mathrm{curl}$, $\mathrm{inc}\, g := \nabla \times g \times \nabla$.

  linear elasticity, defects, linearized curvature...

- divdiv complex:

$$0 \longrightarrow C^\infty \otimes \mathbb{V} \xrightarrow{\text{dev grad}} C^\infty \otimes \mathbb{T} \xrightarrow{\text{sym curl}} C^\infty \otimes \mathbb{S} \xrightarrow{\text{div div}} C^\infty \longrightarrow 0.$$

  $\mathrm{dev}\,\sigma := \sigma - \frac{1}{n}\mathrm{tr}(\sigma) I$ deviator

  adjoint of Hessian complex

There are infinitely many BGG sequences, which may contain operators of arbitrarily high order.

conformal deformation complex

$$0 \longrightarrow C^\infty \otimes \mathbb{R}^3 \xrightarrow{\text{dev def}} C^\infty \otimes (\mathbb{S} \cap \mathbb{T}) \xrightarrow{\text{cot}} C^\infty \otimes (\mathbb{S} \cap \mathbb{T}) \xrightarrow{\text{div}} C^\infty \otimes \mathbb{R}^3 \longrightarrow 0.$$

$\operatorname{cot}(g)$: linearized Cotton-York tensor, third order curl operator
gravitational wave unknowns: transverse-traceless (TT) tensors, i.e., traceless, symmetric, divergence-free

conformal Hessian complex

$$0 \longrightarrow C^\infty \xrightarrow{\text{dev hess}} C^\infty \otimes (\mathbb{S} \cap \mathbb{T}) \xrightarrow{\text{sym curl}} C^\infty \otimes (\mathbb{S} \cap \mathbb{T}) \xrightarrow{\text{div div}} C^\infty \longrightarrow 0.$$

Applications in general relativity:

- Beig, R., & Chruściel, P. T. (2020). On linearised vacuum constraint equations on Einstein manifolds. Classical and Quantum Gravity, 37(21), 215012.

BGG construction (in the above setting):

- Arnold, D. N., & Hu, K. (2021). Complexes from complexes. Foundations of Computational Mathematics, 21(6), 1739-1774.
- Čap, A., & Hu, K. (2023). BGG sequences with weak regularity and applications. Foundations of Computational Mathematics, 1-40.

Sketch of derivation: complexes from complexes

Step 1: connect two (or more) de Rham complexes

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R}^3 \arrow[r, "\mathrm{grad}"] & \mathbb{R}^{3\times 3} \arrow[r, "\mathrm{curl}"] & \mathbb{R}^{3\times 3} \arrow[r, "\mathrm{div}"] & \mathbb{R}^3 \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{R}^3 \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\mathrm{mskw}"] & \mathbb{R}^{3\times 3} \arrow[r, "\mathrm{curl}"] \arrow[ur, "\mathcal{S}"] & \mathbb{R}^{3\times 3} \arrow[r, "\mathrm{div}"] \arrow[ur, "2\,\mathrm{vskw}"] & \mathbb{R}^3 \arrow[r] & 0
\end{tikzcd}$$

$\mathcal{S}u := u^T - \mathrm{tr}(u)I$, bijective

Step 2: eliminate as much as possible

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R}^3 \arrow[r, "\mathrm{grad}"] & \mathbb{S}+\mathbb{K} \arrow[r, "\mathrm{curl}"] & \mathbb{R}^{3\times 3} \arrow[r, "\mathrm{div}"] & \mathbb{R}^3 \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{R}^3 \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\mathrm{mskw}"] & \mathbb{R}^{3\times 3} \arrow[r, "\mathrm{curl}"] \arrow[ur, "\mathcal{S}"] & \mathbb{S}+\mathbb{K} \arrow[r, "\mathrm{div}"] \arrow[ur, "2\,\mathrm{vskw}"] & \mathbb{R}^3 \arrow[r] & 0
\end{tikzcd}$$

$\mathbb{S}$: symmetric matrix, $\mathbb{K}$: skew-symmetric matrix

Step 3: connect rows by zig-zag

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R}^3 \arrow[r, "\mathrm{sym\,grad}"] & \mathbb{S} \arrow[r, "\mathrm{curl}"] & {} \arrow[dl] & & & \\
& & {} \arrow[r, "\mathrm{curl}^T"] & \mathbb{S} \arrow[r, "\mathrm{div}"] & \mathbb{R}^3 \arrow[r] & 0.
\end{tikzcd}$$

Conclusion: the cohomology of the output (elasticity) is isomorphic to the input (de Rham)

Inspired by Bernstein-Gelfand-Gelfand (BGG) resolution (B-G-G 1975, Eastwood 2000, Čap,Slovák,Souček 2001, Arnold,Falk,Winther 2006)

EXAMPLES: FORM-VALUED-FORMS

$$
\begin{tikzcd}
0 \arrow[r] & \mathrm{Alt}^{0,J-1} \arrow[r, "d"] & \mathrm{Alt}^{1,J-1} \arrow[r, "d"] & \cdots \arrow[r, "d"] & \mathrm{Alt}^{n,J-1} \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Alt}^{0,J} \arrow[r, "d"] \arrow[ur, "S^{0,J}"] & \mathrm{Alt}^{1,J} \arrow[r, "d"] \arrow[ur, "S^{1,J}"] & \cdots \arrow[r, "d"] \arrow[ur, "S^{n-1,J}"] & \mathrm{Alt}^{n,J} \arrow[r] & 0
\end{tikzcd}
$$

where $\mathrm{Alt}^{i,J} := \mathrm{Alt}^{i} \otimes \mathrm{Alt}^{J}$

$$
s^{i,J}\mu(v_0, \cdots, v_i)(w_1, \cdots, w_{J-1}) := \sum_{l=0}^{i} (-1)^l \mu(v_0, \cdots, \widehat{v_l}, \cdots, v_i)(v^l, w_1, \cdots, w_{J-1}),
$$

$$
\forall v_0, \cdots, v_i, w_1, \cdots, w_{J-1} \in \mathbb{R}^n.
$$

The S operators have a representation theory background (Lie algebraic cohomology)

3D Examples

$\mathbb{R}$: scalar $\quad$ $\mathbb{V}$: vector $\quad$ $\mathbb{M}$: matrix $\quad$ $\mathbb{S}$: symmetric matrix $\quad$ $\mathbb{T}$: trace-free matrix

$$
\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r, "\mathrm{grad}"] & \mathbb{V} \arrow[r, "\mathrm{curl}"] & \mathbb{V} \arrow[r, "\mathrm{div}"] & \mathbb{R} \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "\mathrm{id}"] & \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "2\,\mathrm{vskw}"] & \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{tr}"] & \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\,\mathrm{mskw}"'] & \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "S"'] & \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "2\,\mathrm{vskw}"'] & \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{R} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\iota"] & \mathbb{V} \arrow[r, "\mathrm{curl}"] \arrow[ur, "\mathrm{mskw}"] & \mathbb{V} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{id}"] & \mathbb{R} \arrow[r] & 0.
\end{tikzcd}
$$

Hessian complex:

$$0 \longrightarrow C^\infty \xrightarrow{\mathrm{hess}} C^\infty(\mathbb{S}) \xrightarrow{\mathrm{curl}} C^\infty(\mathbb{T}) \xrightarrow{\mathrm{div}} C^\infty(\mathbb{V}) \longrightarrow 0.$$

biharmonic equations, plate theory, Einstein-Bianchi system of general relativity

3D EXAMPLES

$\mathbb{R}$: scalar $\mathbb{V}$: vector $\mathbb{M}$: matrix $\mathbb{S}$: symmetric matrix $\mathbb{T}$: trace-free matrix

$$
\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r, "\mathrm{grad}"] & \mathbb{V} \arrow[r, "\mathrm{curl}"] & \mathbb{V} \arrow[r, "\mathrm{div}"] & \mathbb{R} \arrow[r] & 0 \\
{\color{blue}0} \arrow[r] & {\color{blue}\mathbb{V}} \arrow[r, "{\color{blue}\mathrm{grad}}"] \arrow[ur, "\mathrm{id}"] & {\color{blue}\mathbb{M}} \arrow[r, "{\color{blue}\mathrm{curl}}"] \arrow[ur, "2\,\mathrm{vskw}"] & {\color{blue}\mathbb{M}} \arrow[r, "{\color{blue}\mathrm{div}}"] \arrow[ur, "\mathrm{tr}"] & {\color{blue}\mathbb{V}} \arrow[r] & {\color{blue}0} \\
{\color{blue}0} \arrow[r] & {\color{blue}\mathbb{V}} \arrow[r, "{\color{blue}\mathrm{grad}}"] \arrow[ur, "{\color{blue}-\,\mathrm{mskw}}"] & {\color{blue}\mathbb{M}} \arrow[r, "{\color{blue}\mathrm{curl}}"] \arrow[ur, "{\color{blue}S}"] & {\color{blue}\mathbb{M}} \arrow[r, "{\color{blue}\mathrm{div}}"] \arrow[ur, "{\color{blue}2\,\mathrm{vskw}}"] & {\color{blue}\mathbb{V}} \arrow[r] & {\color{blue}0} \\
0 \arrow[r] & \mathbb{R} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\iota"] & \mathbb{V} \arrow[r, "\mathrm{curl}"] \arrow[ur, "\mathrm{mskw}"] & \mathbb{V} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{id}"] & \mathbb{R} \arrow[r] & 0.
\end{tikzcd}
$$

elasticity complex:

$$0 \longrightarrow C^\infty(\mathbb{V}) \xrightarrow{\mathrm{def}} C^\infty(\mathbb{S}) \xrightarrow{\mathrm{inc}} C^\infty(\mathbb{S}) \xrightarrow{\mathrm{div}} C^\infty(\mathbb{V}) \longrightarrow 0.$$

elasticity, defects, metric, curvature

3D EXAMPLES

$\mathbb{R}$: scalar $\mathbb{V}$: vector $\mathbb{M}$: matrix $\mathbb{S}$: symmetric matrix $\mathbb{T}$: trace-free matrix

$$
\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r, "\mathrm{grad}"] & \mathbb{V} \arrow[r, "\mathrm{curl}"] & \mathbb{V} \arrow[r, "\mathrm{div}"] & \mathbb{R} \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "\mathrm{id}"] & \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "2\,\mathrm{vskw}"] & \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{tr}"] & \mathbb{V} \arrow[r] & 0 \\
{\color{blue}0} \arrow[r] & {\color{blue}\mathbb{V}} \arrow[r, "{\color{blue}\mathrm{grad}}"] \arrow[ur, "-\,\mathrm{mskw}"] & {\color{blue}\mathbb{M}} \arrow[r, "{\color{blue}\mathrm{curl}}"] \arrow[ur, "S"] & {\color{blue}\mathbb{M}} \arrow[r, "{\color{blue}\mathrm{div}}"] \arrow[ur, "2\,\mathrm{vskw}"] & {\color{blue}\mathbb{V}} \arrow[r] & {\color{blue}0} \\
{\color{blue}0} \arrow[r] & {\color{blue}\mathbb{R}} \arrow[r, "{\color{blue}\mathrm{grad}}"] \arrow[ur, "{\color{blue}-\iota}"] & {\color{blue}\mathbb{V}} \arrow[r, "{\color{blue}\mathrm{curl}}"] \arrow[ur, "{\color{blue}\mathrm{mskw}}"] & {\color{blue}\mathbb{V}} \arrow[r, "{\color{blue}\mathrm{div}}"] \arrow[ur, "{\color{blue}\mathrm{id}}"] & {\color{blue}\mathbb{R}} \arrow[r] & {\color{blue}0.}
\end{tikzcd}
$$

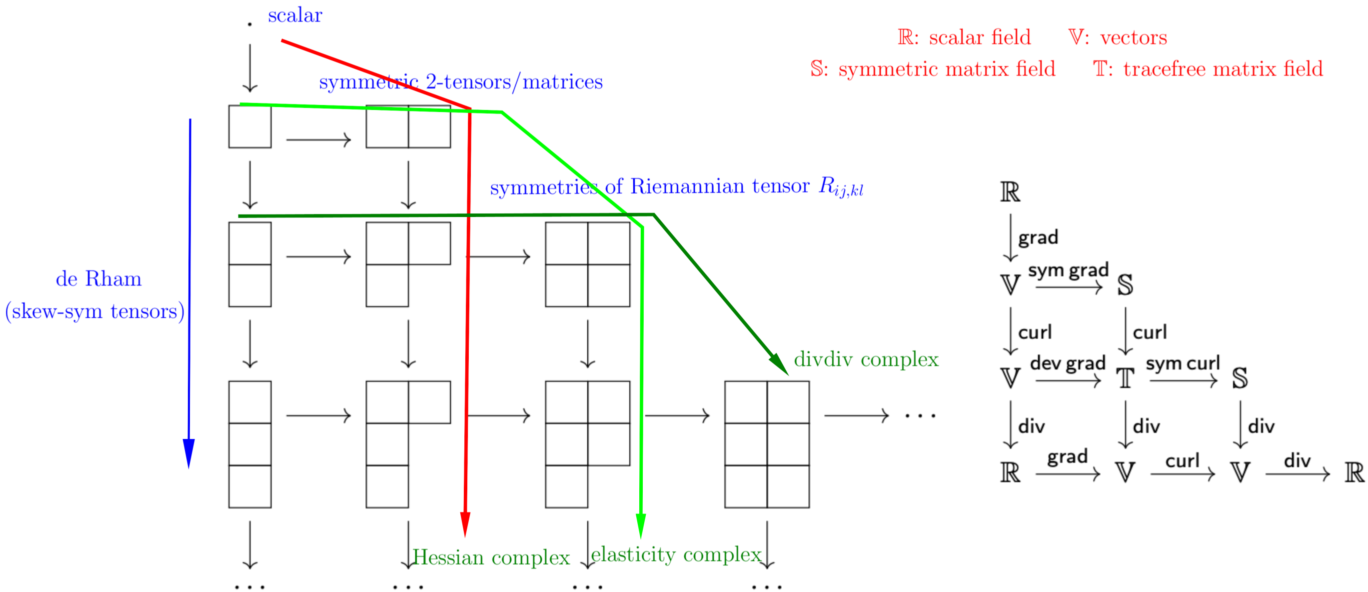
divdiv complex:

$$0 \longrightarrow C^\infty(\mathbb{V}) \xrightarrow{\mathrm{dev\,grad}} C^\infty(\mathbb{T}) \xrightarrow{\mathrm{sym\,curl}} C^\infty(\mathbb{S}) \xrightarrow{\mathrm{div\,div}} C^\infty \longrightarrow 0.$$

plate theory, elasticity

A DIFFERENT PICTURE: HOW TO CHARACTERIZE HIGH-ORDER TENSORS?

Young tableaux



Peter Olver, ’Differential hyperforms’ 1982.

Discretization: ongoing work with Gopalakrishnan, Schöberl.

For each complex, consider various kinds of Sobolev regularity

Two basic classes:

- H^q complex:

$$0 \longrightarrow H^q\Lambda^0 \xrightarrow{d^0} H^{q-1}\Lambda^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} H^{q-n}\Lambda^n \longrightarrow 0,$$

  where $H^q\Lambda^k$ is k-forms with coefficients in H^q.
- $H\Lambda$ complex:

$$0 \longrightarrow H\Lambda^0 \xrightarrow{d^0} H\Lambda^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} H\Lambda^n \longrightarrow 0,$$

  where $H\Lambda^k := \{u \in L^2\Lambda^k : d^k u \in L^2\Lambda^{k+1}\}$.

Sobolev spaces in terms of continuity: for piecewise smooth functions,

$$u \in H^q \iff u \in C^{q-1}$$

$$w \in H\Lambda^k \iff \mathrm{tr}_{\partial T}\, w \text{ continuous}$$

(trace: $\mathrm{tr}_{\partial T} := \iota^*_{\partial T \to T}$, boundary term from integration by parts $(du, v) = (u, \delta v) +$ boundary term)

The cohomology of the above complexes is isomorphic to the smooth de Rham cohomology.

Proof: consequence of Costabel-McIntosh's results for the H^q-complex.

Costabel, M., & McIntosh, A. (2010). On Bogovskiĭ and regularized Poincaré integral operators for de Rham complexes on Lipschitz domains. Mathematische Zeitschrift, 265(2), 297-320.

Examples

H^q complex: $0 \longrightarrow H^3 \xrightarrow{\text{grad}} \left[H^2\right]^3 \xrightarrow{\text{curl}} \left[H^1\right]^3 \xrightarrow{\text{div}} L^2 \longrightarrow 0$

$H\Lambda$ complex: $0 \longrightarrow H^1 \xrightarrow{\text{grad}} H(\text{curl}) \xrightarrow{\text{curl}} H(\text{div}) \xrightarrow{\text{div}} L^2 \longrightarrow 0$

$$\begin{aligned} u \in H^q \quad &\Longleftrightarrow \quad u \in C^{q-1} \\ \boldsymbol{w} \in H(\text{curl}) \quad &\Longleftrightarrow \quad \boldsymbol{w} \times \boldsymbol{n} \text{ continuous (tangential continuity)} \\ \boldsymbol{v} \in H(\text{div}) \quad &\Longleftrightarrow \quad \boldsymbol{v} \cdot \boldsymbol{n} \text{ continuous (normal continuity)} \end{aligned}$$

(infinitely) many combinations, e.g., Stokes complexes

$$0 \longrightarrow H^2 \xrightarrow{\text{grad}} H^1(\text{curl}) \xrightarrow{\text{curl}} \left[H^1\right]^3 \xrightarrow{\text{div}} L^2 \longrightarrow 0,$$

$H^1(\text{curl}) := \{\boldsymbol{u} \in \left[H^1\right]^3 : \text{curl}\, \boldsymbol{u} \in \left[H^1\right]^3\}$ (p.w.smooth $\boldsymbol{u}$: $\boldsymbol{u}$ continuous with curl $\boldsymbol{u}$ continuous);

$$0 \longrightarrow H^1 \xrightarrow{\text{grad}} H_+(\text{curl}) \xrightarrow{\text{curl}} \left[H^1\right]^3 \xrightarrow{\text{div}} L^2 \longrightarrow 0,$$

$H_+(\text{curl}) := \{\boldsymbol{v} \in \boldsymbol{L}^2 : \text{curl}\, \boldsymbol{v} \in \left[H^1\right]^3\}$ ($\boldsymbol{v}$ tangential continuity with all components of curl $\boldsymbol{v}$ continuous).

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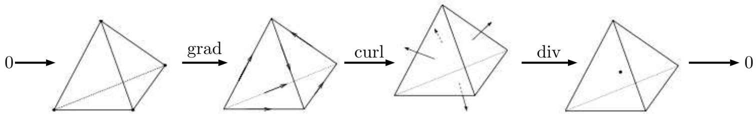
On the canonical finite elements



- degrees of freedom $\cong$ cochains (k-forms on k-cells)
- the incomplete $\mathcal{P}_1$ polynomials

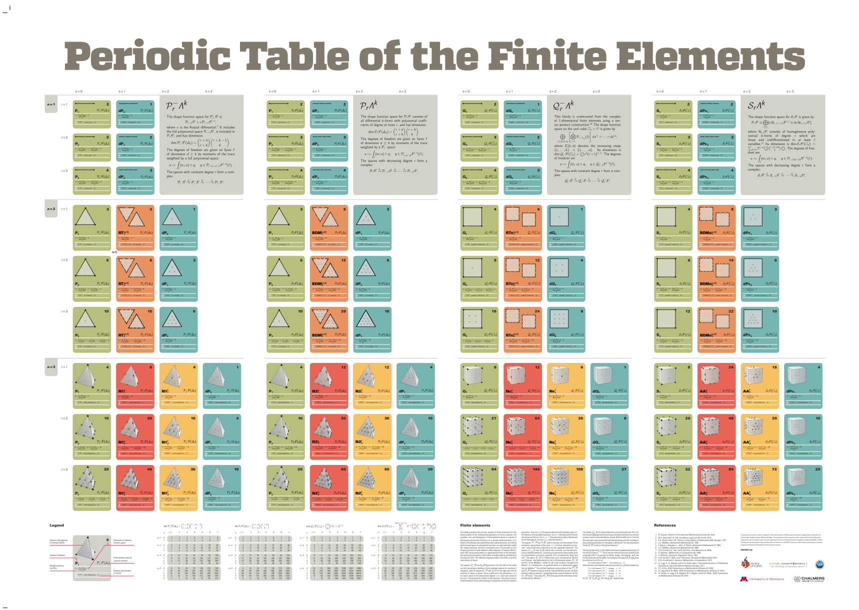
$$0 \longrightarrow \mathcal{P}_1 \xrightarrow{\text{grad}} [\mathcal{P}_0]^3 + \boldsymbol{x} \times [\mathcal{P}_0]^3 \xrightarrow{\text{curl}} [\mathcal{P}_0]^3 + \boldsymbol{x}\mathcal{P}_0 \xrightarrow{\text{div}} \mathcal{P}_0 \longrightarrow 0$$

  extends abstract cochains to piecewise polynomials.
- C^0, tangential continuity, normal continuity, discontinuous, respectively. Subspaces of

$$0 \longrightarrow H^1 \xrightarrow{\text{grad}} H(\text{curl}) \xrightarrow{\text{curl}} H(\text{div}) \xrightarrow{\text{div}} L^2 \longrightarrow 0.$$

- invented individually by Raviart, Thomas, Nédélec in 1970s-1980s.
  Coincide with Whitney forms (Geometric Integration Theory, 1957)

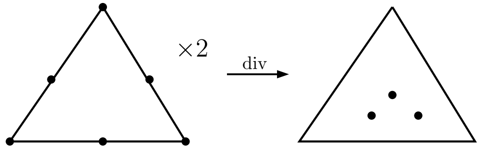
A systematic construction exists, extending to any dimension and any polynomial degree.



Arnold, Logg 2014, SIAM news

Higher continuity: 2D Stokes, a motivating example

- Stokes problem: construct $V_h \subset [H^1]^n$, $Q_h \subset L^2$, such that div $V_h = Q_h$ and inf-sup condition.
  Puzzle of Scott-Vogelius ($[C^0\mathcal{P}_r]^n$-$C^{-1}\mathcal{P}_{r-1}$) : 2D stable for $r \geq 4$, no “singular vertices”; 3D open.



  Scott-Vogelius pair is stable on certain meshes (Arnold-Qin 1992, Qin-S.Zhang 2007, S.Zhang 2008)
- dimension of spline spaces: $\dim(S_r^k(\Delta)) = ??$
  - $k = 1$ in $\mathbb{R}^2$: Billera, algebraic geometry and homological techniques,
  - in general: open.

The two questions are related in a de Rham complex[1]:

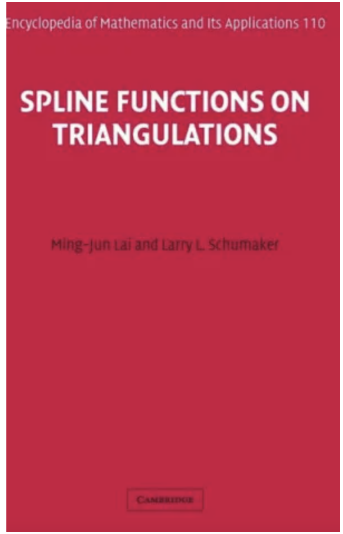
$$0 \longrightarrow C^1 \text{ spline } \xrightarrow{\text{curl}} V_h \xrightarrow{\text{div}} Q_h \longrightarrow 0.$$

For example, a necessary condition of exactness is the dimension count

$$\dim(C^1 \text{ spline }) - \dim(V_h) + \dim Q_h = \dim(\mathcal{N}(\text{curl})) = 1.$$

[1] 2D curl operator maps scalar to vector: $\text{curl}\, u = (-\partial_2 u, \partial_1 u)$. Rotation of grad.

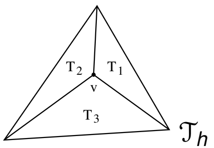
The connection inspires...

$d\Bigg($  $\Bigg)$

The spline literature provides a good start for complexes (as 0-forms). However, completing the sequences is nontrivial. Below we review some general ideas and tools.

General strategies: use or avoid supersmoothness

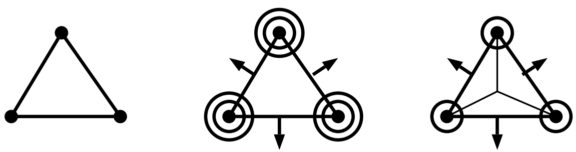
Supersmoothness is a subtle issue in constructing scalar (simplicial) splines:
Piecewise smooth functions + corners lead to automatic higher continuity .



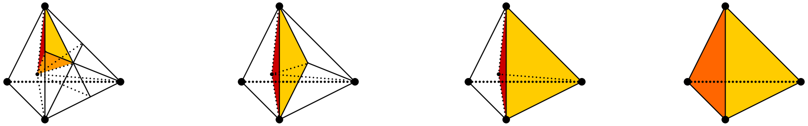
$$C^1(\mathcal{T}_h) \implies C^2(v)$$

- Sorokina, T. (2010). Intrinsic supersmoothness of multivariate splines. Numerische Mathematik, 116, 421-434.
- Shekhtman, B., & Sorokina, T. (2015). Intrinsic Supermoothness. Journal of Concrete & Applicable Mathematics, 13.
- Floater, M. S., & Hu, K. (2020). A characterization of supersmoothness of multivariate splines. Advances in Computational Mathematics, 46(5), 70.

Consequence on the construction of FEs/splines: either use supersmoothness as degrees of freedom, or use macroelement split to avoid supersmoothness.



C^0 $\qquad$ C^1 $\qquad$ C^1



Refinements of a tetrahedron relative to one face: Worsey-Piper, Worsey-Farin, Alfeld, no split.
(each 2D face: Powell-Sabin, Clough-Tocher/Alfeld, no split, no split)

Lai & Schumaker 2007, *Spline functions on triangulations*. CUP

General experience: a complex usually has various versions with different subdivision. Finer refinement requires less supersmoothness (rigorous theorems).

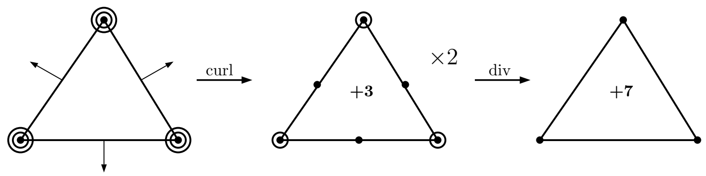
See Guzmán talk.

Back to the Stokes example

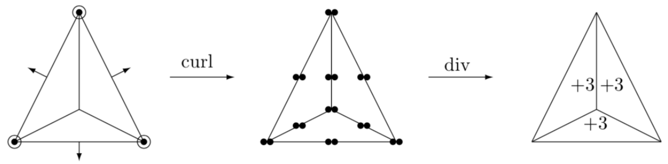
smoothness of scalar splines propagate in the complexes

$$0 \longrightarrow H^2 \xrightarrow{\text{grad}} [H^1]^2 \xrightarrow{\text{rot}} L^2 \longrightarrow 0.$$

Supersmoothness as DoFs: Falk-Neilan 2013, leading to a new Stokes pair



Clough-Tocher split: Arnold-Qin 1992 (last two spaces), Christiansen-KH 2018

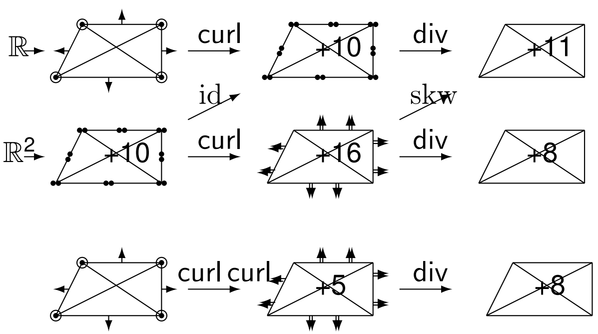


Guzmán-Lischke-Neilan 2019: Powell-Sabin split (Stokes pair: S.Zhang 2008)

Several other examples exist.

complexes relate existing results and lead to new.

2D Stokes (2013)	Argyris (1960s)	Falk-Neilan (2013)
2D Stokes (2018)	Clough-Tocher (1965)	Arnold-Qin (1992)
2D Stokes (2020)	Powell-Sabin (1977)	Zhang (2011)
2D elasticity (2002)	Argyris (1960s)	Arnold-Winther (2002), Hu-Zhang (2014)
2D elasticity (2022)	Clough-Tocher (1965)	Johnson-Mercier (1978)
3D Stokes (2015)	Zěníšek (1973)	Neilan (2015)
3D Stokes (2020)	Alfeld (1984)	Zhang (2005)
3D Stokes (2022)	Worsey-Farin (1988)	Zhang (2011)
3D elasticity (2020)	Alfeld (1984)	Christiansen-Gopalakrishnan-Guzman-KH (2020)



a diagram connecting splines (Lai-Schumaker book), fluids (Arnold-Qin 1992) and elasticity (Johnson-Mercier 1978)

General strategies: dimension count

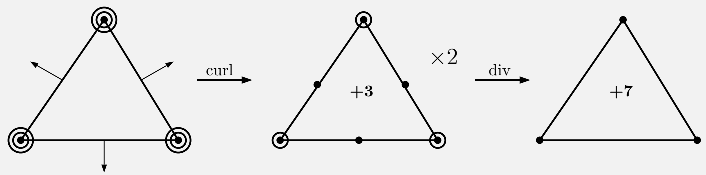
$$\cdots \longrightarrow V^{k-1} \xrightarrow{d^{k-1}} V^k \xrightarrow{d^k} V^{k+1} \longrightarrow \cdots,$$

a necessary condition for exactness is

$$\sum_{j=0}^{n} \dim(V^j) = 0.$$

Dimension count gives a hint guiding the construction.

Example: Falk-Neilan complex



dimension count is $6V + E \longrightarrow 2(3V + E + 3T) \longrightarrow V + 7E$. We have

$$(6V + E) - 2(3V + E + 3T) + (V + 7T) = V - E + T = 1 = \dim \mathcal{N}(\text{curl}),$$

V: number of vertices, E: number of edges, T: number of triangles, and Euler characteristic used.

General strategies: Poincaré operators

The first step to construct FE spaces is to have polynomial exact sequences on each simplex. How?

Poincaré operators:

$$\cdots \rightleftarrows V^{i-1} \underset{P^i}{\overset{d^{i-1}}{\rightleftarrows}} V^i \underset{P^{i+1}}{\overset{d^i}{\rightleftarrows}} V^{i+1} \rightleftarrows \cdots$$

$P^k : V^k \mapsto V^{k-1}$, satisfying null-homotopy property

$$d^{k-1}P^k + P^{k+1}d^k = I_{V^k},$$

Existence of Poincaré operators implies exactness:

$$du = 0 \quad \Longrightarrow \quad u = (dP + Pd)u = d(Pu).$$

Smooth de Rham complex: explicit forms known in differential manifold books (proof of Poincaré lemma, integral operators along a path). Choosing path $\gamma(t) = tx$, for $t \in [0, 1]$, connecting 0 (base point) and x:

$$\mathfrak{p}^1 u = \int_0^1 u_{tx} \cdot x dt, \quad \mathfrak{p}^2 v = \int_0^1 t v_{tx} \times x dt, \quad \mathfrak{p}^3 w = \int_0^1 t^2 w_{tx} x dt.$$

Sobolev de Rham complex: averaging the base point

Costabel, M., & McIntosh, A. (2010). On Bogovskiĭ and regularized Poincaré integral operators for de Rham complexes on Lipschitz domains. Mathematische Zeitschrift, 265(2), 297-320.

Smooth and Sobolev BGG complexes: follow the BGG steps

- Christiansen, S. H., Hu, K., & Sande, E. (2020). Poincaré path integrals for elasticity. Journal de Mathématiques Pures et Appliquées, 135, 83-102.
- Čap, A., & Hu, K. (2023). Bounded Poincaré operators for twisted and BGG complexes. Journal de Mathématiques Pures et Appliquées, 179, 253-276.

The above operators preserve polynomial calsses, implying exactness of polynomial complexes, e.g.,

the $\mathcal{P}_r$ family: $\cdots \longrightarrow \mathcal{P}_{r-(k-1)}\Lambda^{k-1} \xrightarrow{d^{k-1}} \mathcal{P}_{r-k}\Lambda^k \xrightarrow{d^k} \mathcal{P}_{r-(k+1)}\Lambda^{k+1} \longrightarrow \cdots$

the $\mathcal{P}_r^-$ family: $\cdots \longrightarrow \mathcal{P}_r\Lambda^{k-1} + P^k\mathcal{P}_r\Lambda^k \xrightarrow{d^{k-1}} \mathcal{P}_r\Lambda^k + P^{k+1}\mathcal{P}_r\Lambda^{k+1} \xrightarrow{d^k} \mathcal{P}_r\Lambda^k + P^{k+1}\mathcal{P}_r\Lambda^{k+1} \longrightarrow \cdots$

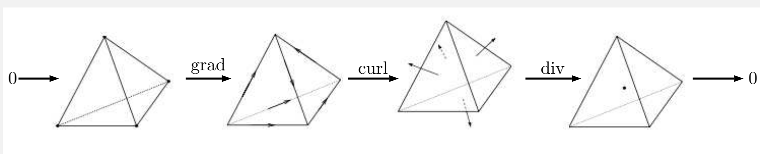
General strategies: Geometric decomposition and bubble complexes

Lagrange element ($\mathcal{P}_r C^0$) =
cell bubbles (interior BB basis, vanishing on all faces)+ face bubbles (face BB basis, vanishing on all edges)
+ edge bubbles(vanishing at vertices) + vertex modes

Similar decompositions hold for vector/tensor finite elements
(bubbles have vanishing *trace*, e.g., $H(div)$-bubbles have vanishing normal components).

Bubble spaces should form exact sequences.

Remark: Cohomology is only carried by the lowest order spaces.



Idea 1. Explicit forms of bubbles can be used to prove exactness.

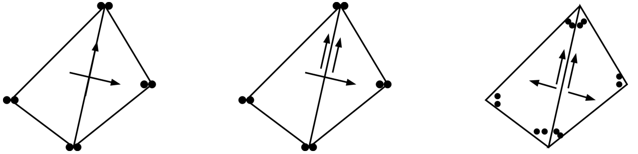
e.g., $H(\mathrm{div})$ bubbles = 'real' bubbles (all components vanish) + fields tangent to the faces.

$$\mathcal{B}_r := \lambda_0\lambda_1\lambda_2 \left[\mathcal{P}_{r-1}\right]^2 + \lambda_i\lambda_j\mathcal{P}_r \boldsymbol{t}_{ij}$$

Then show $\mathrm{div} : \lambda_0\lambda_1\lambda_2 \left[\mathcal{P}_{r-3}\right]^2 + \lambda_i\lambda_j\mathcal{P}_{r-2}\boldsymbol{t}_{ij} \to \mathcal{P}_{r-1}/\mathbb{R}$ is surjective.

Idea 2. Each tensor Lagrange/BB basis has a normal-tangential decomposition.

e.g., for $H(\mathrm{div})$, tangential bases are all bubbles. Normal bases on different cells should be matched.



continuous, continuous normal, and discontinuous.

For Idea 1, see, e.g.,

- Hu, J. (2015). Finite element approximations of symmetric tensors on simplicial grids in $\mathbb{R}^n$: The higher order case. Journal of Computational Mathematics, 283-296.

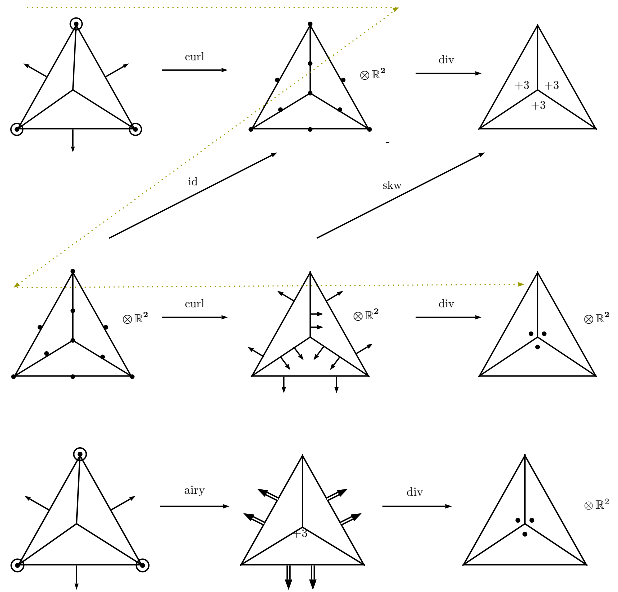
For Idea 2, see, e.g.,

- Chen, L., & Huang, X. (2021). Geometric decompositions of div-conforming finite element tensors. arXiv preprint arXiv:2112.14351, 2.
- Hu, J., Hu, K., & Zhang, Q. (2022). Partially Discontinuous Nodal Finite Elements for $H(\mathrm{curl})$ and $H(\mathrm{div})$. Computational Methods in Applied Mathematics, 22(3), 613-629.

General strategies: Diagram chase for BGG complexes

BGG complexes can be derived from de Rham. One can mimic the construction for FE/spline spaces.

Example:



Challenge: To match the spaces, the input rows should have different regularity.
This approach requires a lot of spline complexes with various kinds of continuity as input.

General strategies: Tensor product construction

So far we focused on 'splines on triangulation'. On cubical meshes, tensor product is a useful tool.

Idea: for 0-forms (scalars), approximate $f(x, y)$ by $u(x)v(y)$. Then extend this to a complex.

Example: $\mathcal{S}^{p_1,p_2}_{r_1,r_2}$: degree p_1 in x, p_2 in y; smoothness r_1 in x, r_2 in y

$$0 \longrightarrow \mathcal{S}^{p_1,p_2}_{r_1,r_2} \xrightarrow{\text{curl}} \begin{pmatrix} \mathcal{S}^{p_1,p_2-1}_{r_1,r_2-1} \\ \mathcal{S}^{p_1-1,p_2}_{r_1-1,r_2} \end{pmatrix} \xrightarrow{\text{div}} \mathcal{S}^{p_1-1,p_2-1}_{r_1-1,r_2-1} \longrightarrow 0$$

partial derivatives decrease degree and regularity by 1.

- Buffa, A., Rivas, J., Sangalli, G., & Vázquez, R. (2011). Isogeometric discrete differential forms in three dimensions. SIAM Journal on Numerical Analysis, 49(2), 818-844.
- Christiansen, S. H. (2009). Foundations of finite element methods for wave equations of Maxwell type. Springer.

de Rham complexes: Arnold,Boffi,Bonizzoni 2013

$$\mathcal{Q}^-_r\Lambda^k(I^n) = \bigoplus_{\sigma\in\Sigma(k;n)} \left[\bigotimes_{i=1}^{n} \mathcal{P}_{r-\delta_{i,\sigma}}(I)\right] dx^{\sigma_1}\wedge\cdots\wedge dx^{\sigma_k},$$

where

$$\delta_{i,\sigma} = \begin{cases} 1, & i \in \{\sigma_1, \cdots, \sigma_k\}, \\ 0, & \text{otherwise.} \end{cases}$$

higher form degree corresponds to lower polynomial degree (in a delicate way).

For BGG complexes, one can either diagram chase or discover a general pattern of indices.

- Bonizzoni, F., Hu, K., Kanschat, G., & Sap, D. (2024). Discrete tensor product BGG sequences: splines and finite elements. Mathematics of Computation.
- Arf, J., & Simeon, B. (2021). Structure-preserving discretization of the Hessian complex based on spline spaces. arXiv preprint arXiv:2109.05293.

Examples in 2D: Input: Buffa,Rivas,Sangalli,Vázquez 2011 with various choices of indices

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{S}^{p_1,p_2}_{r_1,r_2} \arrow[r, "\mathrm{grad}"] & \begin{pmatrix} \mathcal{S}^{p_1-1,p_2}_{r_1-1,r_2} \\ \mathcal{S}^{p_1,p_2-1}_{r_1,r_2-1} \end{pmatrix} \arrow[r, "\mathrm{rot}"] & \mathcal{S}^{p_1-1,p_2-1}_{r_1-1,r_2-1} \arrow[r] & 0 \\
0 \arrow[r] & \begin{pmatrix} \mathcal{S}^{p_1-1,p_2}_{r_1-1,r_2} \\ \mathcal{S}^{p_1,p_2-1}_{r_1,r_2-1} \end{pmatrix} \arrow[r, "\mathrm{grad}"] \arrow[ur, "\mathrm{I}"] & \begin{pmatrix} \mathcal{S}^{p_1-2,p_2}_{r_1-2,r_2} & \mathcal{S}^{p_1-1,p_2-1}_{r_1-1,r_2-1} \\ \mathcal{S}^{p_1-1,p_2-1}_{r_1-1,r_2-1} & \mathcal{S}^{p_1,p_2-2}_{r_1,r_2-2} \end{pmatrix} \arrow[r, "\mathrm{rot}"] \arrow[ur, "\mathrm{sskw}"] & \begin{pmatrix} \mathcal{S}^{p_1-2,p_2-1}_{r_1-2,r_2-1} \\ \mathcal{S}^{p_1-1,p_2-2}_{r_1-1,r_2-2} \end{pmatrix} \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{S}^{p_1-1,p_2-1}_{r_1-1,r_2-1} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\mathrm{mskw}"] & \begin{pmatrix} \mathcal{S}^{p_1-2,p_2-1}_{r_1-2,r_2-1} \\ \mathcal{S}^{p_1-1,p_2-2}_{r_1-1,r_2-2} \end{pmatrix} \arrow[r, "\mathrm{rot}"] \arrow[ur, "S"] & \mathcal{S}^{p_1-2,p_2-2}_{r_1-2,r_2-2} \arrow[r] & 0.
\end{tikzcd}
$$

This only works for form-valued forms (Hessian, elasticity, divdiv complexes), not the conformal complexes!

If the indices of the skew-symmetric part match, then the trace does not; and vice versa. There seem to be fundamental challenges in discretizing $\mathbb{S} \cap \mathbb{T}$ matrices.

State-of-the-art and open questions

Ultimate goal: any complex (de Rham, BGG), any dimension, any form degree, any continuity, any polynomial degree

de Rham complexes - what has been done?

	dimension	form degree	continuity	polynomial degree
classical splines	mostly 2 or 3	0	high	low in most literature
classical FEEC, periodic table	any	any	low	any
smoother de Rham (Stokes)	mostly 2 or 3	any	some high continuity	any
Chen-Huang, Hu-Lin-Wu	2, 3	any	any	any
tensor product	any	any	any (certain pattern)	any

So far, only ‘classical FEEC, periodic table’ and tensor product construction are relatively systematic theories. (low continuity and cubical grids significantly simplifies the question)

cohomology is only known for the classical FEEC complexes and some special Stokes complexes.

BGG complexes - what has been done?

	what complexes	dimension	continuity
	Hess, elas, divdiv	2, 3	low
	Hess, elas, divdiv	2	any
tensor product	form-valued-forms	any	any (cerntain patterns)
KH-Lin-Shi 2023	conformal deformation	3	low

QUESTIONS

In principle, any question for scalar splines can be asked for vector- or tensor-spaces. Example:

- dimension
- supersmoothness
- basis and fast algorithms
- Bernstein-Bézier techniques (Alfeld-Sorokina 2016)
- ...

New questions raised in complexes:

- construction
- proving cohomology
- constructing commuting quasi-interpolations (‘bounded cochain projections’ in FEEC term)
- construccting local exact sequences (find bubbles which are exact in *top dimension*)
- ...

A concrete example: $H(\mathrm{div}, \mathbb{S})$ from linear elasticity

symmetric matrix field σ, each component σ_{ij} is p.w. polynomial of degree k, $\sigma \cdot n$ continuous across boundary of cells (n is the normal vector).

Dimension? Likely C^0 supersmoothness at vertices? ...

Table of Contents

Generalizing finite elements: back to de Rham's currents

Question: The Whitney forms are elegant and useful. Are there tensor/BGG versions?

Use currents (measures, Dirac delta), rather than functions.

Geometric Measure Theory , graphics
(Codimensional geometry: A point cloud represents a probability measure; curve cloud, surface cloud...)

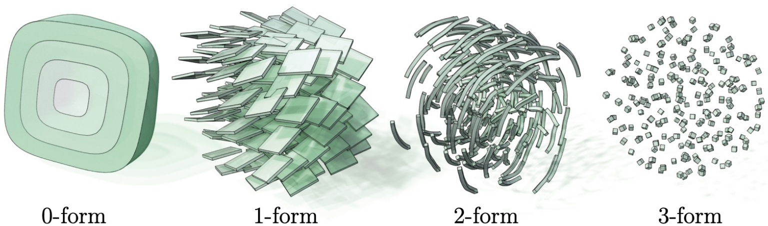
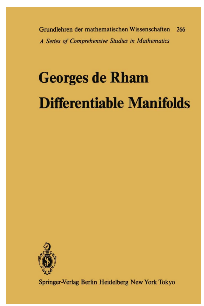
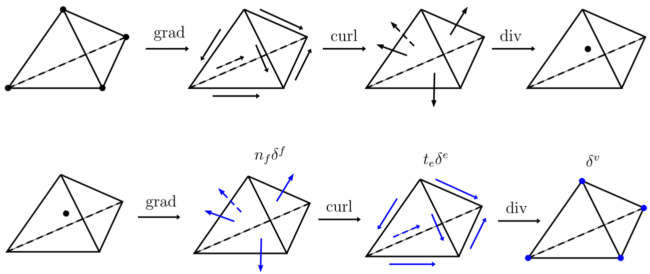


Figure 2.4 Differential k-forms can be represented by clouds of codimension-k geometries.

Exterior Calculus in Graphics: Course Notes for a SIGGRAPH 2023 Course. Wang, S., Nabizadeh, M. S., & Chern, A. (2023)



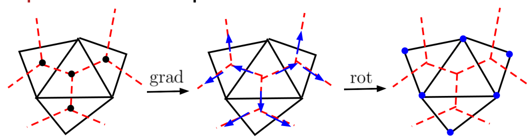
(FINITE ELEMENT VERSIONS: DISTRIBUTIONAL ELEMENTS (CURRENTS))



Braess, Schöberl 2008

Perspectives:

- Finite Element perspective: dual, complex of degrees of freedom
- Discrete Exterior Calculus perspective: complex on dual meshes

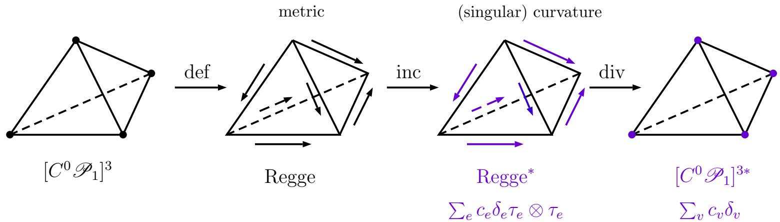


- Fluid perspective: point vortex, vortex lines... (delta on codim 2)
  (V.I.Arnold,B.Khesin, Topological methods in hydrodynamics)

- Applications: equilibrated residual error estimators (Braess, Schöberl 2008)
- Cohomologies, analysis: Licht 2017 (double complex)

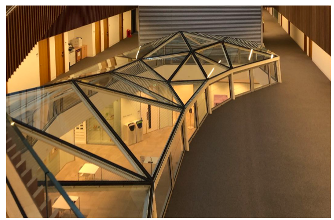
3D ELASTICITY COMPLEX: REGGE CALCULUS

Christiansen 2011: Regge calculus = finite elements



$$\begin{tikzcd} \mathrm{Alt}^{0,1} \arrow[r] & \mathrm{Alt}^{1,1} \arrow[r] & {} \arrow[dl] & \\ & {} \arrow[r] & \mathrm{Alt}^{2,2} \arrow[r] & \mathrm{Alt}^{3,2} \end{tikzcd}$$

Regge calculus: Metric given by edge lengths; curvature as angle deficit (quantum relativity, discrete geometry).

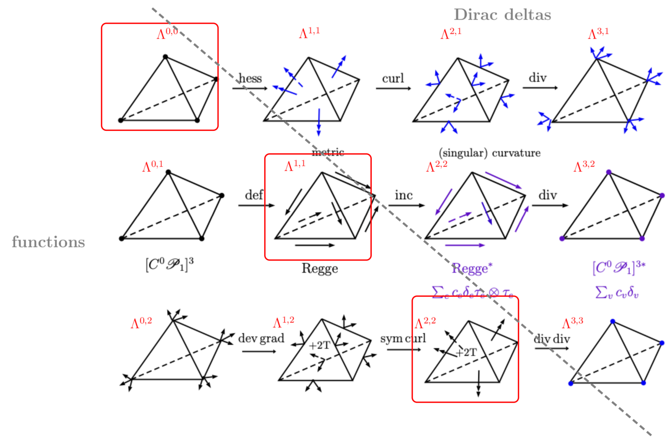


Regge finite element: Metric: p.w. constant sym matrices, $\int_e t_e \cdot g \cdot t_e$ as dofs. Curvature: distributional.

nD: Lizao Li (2018 Minnesota thesis), nonlinear curvature with Regge elements (Berchenko-Kogan,Gawlik 2022, Gopalakrishnan,Neunteufel,Schöberl,Wardetzky 2022, Gawlik,Neunteufel 2023)

Canonical construction in 3D (KH-Lin-Zhang 2023)

$$
\begin{array}{ccccccc}
\mathrm{Alt}^{0,0} & \longrightarrow & & & & & \\
 & \swarrow & & & & & \\
 & \longrightarrow & \mathrm{Alt}^{1,1} & \longrightarrow & \mathrm{Alt}^{2,1} & \longrightarrow & \mathrm{Alt}^{3,1} \\
\mathrm{Alt}^{0,1} & \longrightarrow & \mathrm{Alt}^{1,1} & \longrightarrow & & & \\
 & & & \swarrow & & & \\
 & & & \longrightarrow & \mathrm{Alt}^{2,2} & \longrightarrow & \mathrm{Alt}^{3,2} \\
\mathrm{Alt}^{0,2} & \longrightarrow & \mathrm{Alt}^{1,2} & \longrightarrow & \mathrm{Alt}^{2,2} & \longrightarrow & \\
 & & & & & \swarrow & \\
 & & & & & \longrightarrow & \mathrm{Alt}^{3,3}
\end{array}
$$

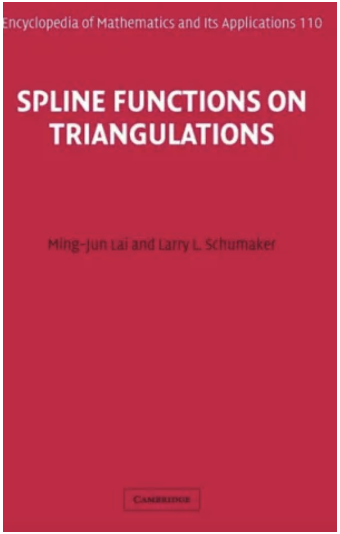


Patterns for discretization for $\mathrm{Alt}^i \otimes \mathrm{Alt}^j$ (j-form-valued i-forms):

- cohomology is correct, isomorphic to continuous version
- $i \leq j$: functions, $i > j$: Dirac deltas, transition happens at BGG zig-zag
- (i, j) dual to (j, i); (i, j) dual to $(n - i, n - j)$, Hessian complex dual to divdiv, elasticity ‘self-adjoint’
- function part ($i \leq j$): j-form-valued i-forms discretized on i-cells attaching a j-form to an i-cochain
- delta part ($i > j$): j-form-valued i-forms means attaching j-forms to dual i-cells

A periodic table for tensors in progress.

SUMMARY

d  $=?$

Encyclopedia of Mathematics and Its Applications 110
SPLINE FUNCTIONS ON TRIANGULATIONS
Ming-Jun Lai and Larry L. Schumaker
CAMBRIDGE