Topology-preserving computation for ideal magnetic relaxation

-Why is helicity-preservation important for MHD computation?-

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Ideal magnetic relaxation

Fundamental question: given initial data, what does the plasma system evolve to?
(whether to steady state, properties of steady state etc.) related to heating of solar corona.

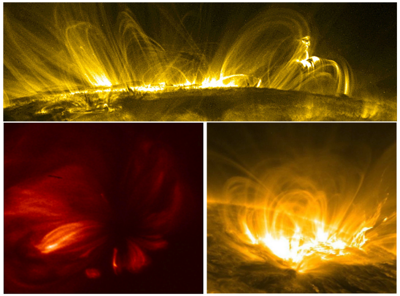
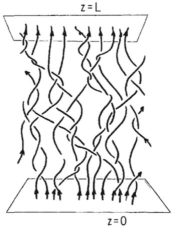


Figure: Kalman J. Knizhnik PhD thesis

Parker hypothesis: for "almost any initial data", *the magnetic field develops tangential discontinuities (current sheet) during the relaxation to static equilibrium.* (Still Open)



Magnetic relaxation is also an important tool for investigating plasma equilibria and is often used to study the magnetic configurations in fusion devices and astrophysical plasmas.

Magnetohydrodynamics (MHD): macroscopic description of plasma, an incompressible model

$$
\begin{aligned}
\partial_t \boldsymbol{u} - \boldsymbol{u} \times (\nabla \times \boldsymbol{u}) - R_e^{-1} \Delta \boldsymbol{u} - s\boldsymbol{j} \times \boldsymbol{B} + \nabla P &= \boldsymbol{f} && \text{momentum equation,} \\
\boldsymbol{j} - \nabla \times \boldsymbol{B} &= \boldsymbol{0} && \text{Ampere's law,} \\
\partial_t \boldsymbol{B} + \nabla \times \boldsymbol{E} &= \boldsymbol{0} && \text{Faraday's law,} \\
R_m^{-1} \boldsymbol{j} - (\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}) &= \boldsymbol{0} && \text{Ohm's law,} \\
\nabla \cdot \boldsymbol{B} &= 0 && \text{Gauss law,} \\
\nabla \cdot \boldsymbol{u} &= 0,
\end{aligned}
$$

initial conditions $\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x})$, $\quad \boldsymbol{B}(\boldsymbol{x}, 0) = \boldsymbol{B}_0(\boldsymbol{x})$,

boundary conditions on $\partial\Omega$: $\boldsymbol{u} = \boldsymbol{0}$, $\quad \boldsymbol{B} \cdot \boldsymbol{n} = 0$, $\quad \boldsymbol{E} \times \boldsymbol{n} = \boldsymbol{0}$.

Three nonlinear terms:

fluid advection $-\boldsymbol{u} \times (\nabla \times \boldsymbol{u})$ (in the vorticity form)
Lorentz force $-s\boldsymbol{j} \times \boldsymbol{B}$
magnetic advection $-\nabla \times (\boldsymbol{u} \times \boldsymbol{B})$

In particular, we are interested in zero magnetic diffusion, nonzero fluid diffusion ($R_m = \infty$, $R_e < \infty$).

Energy structures of MHD

Energy dissipation or conservation:

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}\|_0^2 + \frac{S}{2}\frac{d}{dt}\|\boldsymbol{B}\|_0^2 + R_e^{-1}\|\nabla \boldsymbol{u}\|_0^2 + SR_m^{-1}\|\boldsymbol{j}\|_0^2 = (\boldsymbol{f}, \boldsymbol{u}),$$

and hence

$$\max_{0\le t\le T}\left(\|\boldsymbol{u}\|_0^2 + S\|\boldsymbol{B}\|_0^2\right) + R_e^{-1}\int_0^T \|\nabla \boldsymbol{u}\|_0^2 \, \mathrm{d}\tau + 2SR_m^{-1}\int_0^T \|\boldsymbol{j}\|_0^2 \, \mathrm{d}\tau$$
$$\le \|\boldsymbol{u}_0\|_0^2 + S\|\boldsymbol{B}_0\|_0^2 + R_e \int_0^T \|\boldsymbol{f}\|_{-1}^2 \, \mathrm{d}\tau.$$

With $\boldsymbol{f} = 0$, $R_m^{-1} = 0$, total energy is non-increasing. However, some key information is not clear:

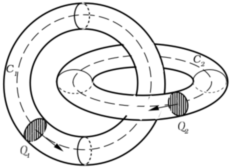
- whether the total energy decays to zero?
- how does the total energy split into the the fluid part ($\|\boldsymbol{u}\|^2$) and the magnetic part ($S\|\boldsymbol{B}\|^2$)?

HELICITY: FINE STRUCTURES

Magnetic helicity: for magnetic potential $\boldsymbol{A}$ satisfying $\nabla \times \boldsymbol{A} = \boldsymbol{B}$,

$$\text{magnetic helicity } \mathcal{H}_m := \int_\Omega \boldsymbol{A} \cdot \boldsymbol{B}\, dx$$

- Idea started from Helmholtz & Kelvin. MHD: Woltjer's invariant, ideal fluid: Moffatt (giving the name).
- characterizing linking/knottedness of $\boldsymbol{B}$. Example: $\mathcal{H}_\xi = 2I(C_1, C_2)Q_1 \cdot Q_2$, where I is the Gauss linking number (topological quantity, =1 in the figure below).



Arnold, Khesin, *Topological methods in hydrodynamics*, 1999

Helicity = averaging asymptotic linking number (continuum version of linked tubes) (V.I. Arnold)

Cross helicity:

$$\text{cross helicity } \mathcal{H}_c := \int_\Omega \boldsymbol{u} \cdot \boldsymbol{B}\, dx$$

linking of vorticity and magnetic fields

More about helicity

Arnold inequality (V.I. Arnold 1974): lower bound for *magnetic* energy

$$\left|\int \boldsymbol{A}\cdot\boldsymbol{B}\,dx\right| \leq C\int |\boldsymbol{B}|^2\,dx.$$

Proof. Cauchy-Schwarz $|\int \boldsymbol{A}\cdot\boldsymbol{B}\,dx| \leq \|\boldsymbol{A}\|_{L^2}\|\boldsymbol{B}\|_{L^2}$ + Poincaré $\|\boldsymbol{A}\|_{L^2} \leq C\|\nabla\times\boldsymbol{A}\|_{L^2}$.

Differential form point of view: $\boldsymbol{A}$: 1-form, $\boldsymbol{B}$: 2-form

$$\underbrace{\int \boldsymbol{A}\wedge\boldsymbol{B}}_{\text{helicity, topology}} \leq C\underbrace{\int \boldsymbol{B}\wedge *\boldsymbol{B}}_{\text{energy, geometry}}$$

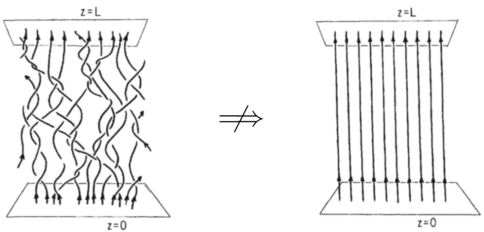
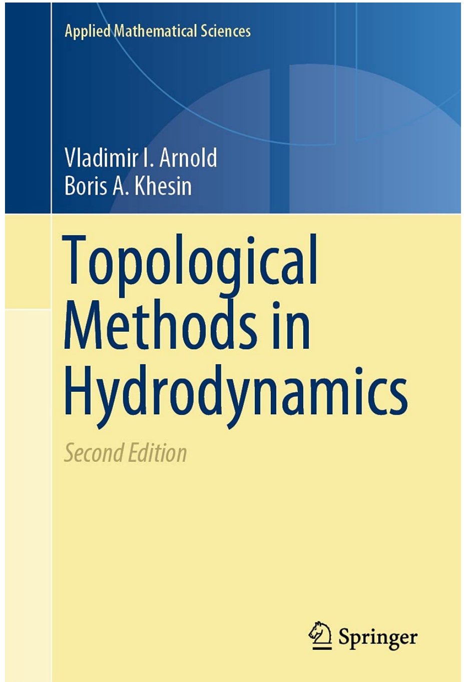


Fig: Pontin, Hornig, Living Rev. Sol. Phys. 2020.

knots are topological barriers that prevent energy from dissipation


Applied Mathematical Sciences
Vladimir I. Arnold
Boris A. Khesin
Topological Methods in Hydrodynamics
Second Edition
Springer

Magnetic and cross helicity are conservative for ideal MHD ($R_e^{-1} = R_m^{-1} = 0$).

$$\frac{d}{dt}\int \boldsymbol{A}\cdot\boldsymbol{B}\,dx = 0, \quad \frac{d}{dt}\int \boldsymbol{u}\cdot\boldsymbol{B}\,dx = 0.$$

The conservation of magnetic helicity holds even with fluid diffusion.

Proof Advection of magnetic fields:

$$\boldsymbol{B}_t - \nabla\times(\boldsymbol{u}\times\boldsymbol{B}) = 0.$$

Then

$$\frac{d}{dt}\int \boldsymbol{A}\cdot\boldsymbol{B} = 2\frac{d}{dt}\int \boldsymbol{A}\cdot\nabla\times(\boldsymbol{u}\times\boldsymbol{B}) \overset{*}{=} 2\int \nabla\times\boldsymbol{A}\cdot(\boldsymbol{u}\times\boldsymbol{B}) = 2\int \boldsymbol{u}\cdot(\boldsymbol{B}\times\boldsymbol{B}) = 0.$$

$*$: integral by parts with proper vanishing boundary conditions.

The conservation of magnetic helicity $\mathcal{H}_m$ does not depend on $\boldsymbol{u}$.

Consequences: e.g., consider a system with fluid diffusion ($R_e < \infty$), without magnetic diffusion ($R_m = \infty$). Energy may decay (due to fluid diffusion), but has a lower bound (by magnetic helicity, which remains constant). So (topologically) nontrivial initial data cannot evolve to a trivial stationary state. This provides a topological constraint for ideal magnetic relaxation.

Computational approach

Evolve the MHD system ($R_e < \infty$, $R_m = \infty$) for a long time.

A *"simplified"* model: diffusive + magnetic helicity conserved

$$\begin{aligned} \boldsymbol{B}_t - \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) &= 0, \\ \boldsymbol{j} &= \nabla \times \boldsymbol{B}, \\ \boldsymbol{u} &= \tau \boldsymbol{j} \times \boldsymbol{B}. \end{aligned}$$

Energy decay

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{B}\|^2 = -\tau\|\boldsymbol{B} \times \boldsymbol{j}\|^2.$$

Computational challenges:

> *Direct computational assessment of Parker's hypothesis brings a number of challenges. Foremost among these is the requirement to* **precisely maintain the magnetic topology during the simulated evolution***, i.e., precisely maintain the magnetic field line mapping between the two line-tied boundaries. ... In the following sections two methods are described which seek to mitigate against these difficulties. However, in all cases the* **representation of current singularities remains problematic***....*

– *The Parker problem: existence of smooth force-free fields and coronal heating*, Pontin, Hornig, Living Rev. Sol. Phys. 2020.

De Rham complex-based method naturally satisfies all the conditions!

- helicity-preservation, discrete Arnold inequality (topological barriers),
- Raviart-Thomas (RT) FE space allows tangential discontinuity, $\nabla\times$RT allows current sheets.

A HELICITY-PRESERVING METHOD

Existing numerical methods for magnetic relaxation: Lagrange method, issues with mesh deformation.
Mimetic methods for Lagrangian relaxation of magnetic fields, S.Candelaresi, D.Pontin, G.Hornig, SIAM Journal on Scientific Computing (2014).

Structure-preserving discretization for MHD:

- energy conservation: e.g., Armero, Simo 1996
- $\nabla \cdot \boldsymbol{B} = 0$: e.g., Brackbill, Barnes 1980
- helicity conservation: less attention, Liu,Wang 2004 (axisymmetric MHD flow, finite difference methods); Kraus,Maj 2017 (DEC, variational integrator), Sullivan 2018 ('Lattice hydrodynamics').

Helicity-preserving finite element for NS: Rebholz 2007; Zhang, Palha, Gerritsma, Rebholz 2022 (dual field approach).

Helicity-preserving finite element for MHD: KH, Lee, Xu 2021; Gawlik, Gay-Balmaz 2022; Laakmann, KH, Farrell 2023 (Hall MHD).

The numerics below are based on the projection approach (Rebholz 2007, KH, Lee, Xu 2021).

DE RHAM COMPLEX

$$\cdots \longrightarrow \Lambda^{k-1} \xrightarrow{d^{k-1}} \Lambda^k \xrightarrow{d^k} \Lambda^{k+1} \longrightarrow \cdots$$

Λ^k : differential k-forms, d^k : exterior derivatives

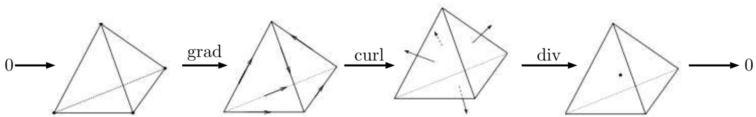
3D in coordinates

$$0 \longrightarrow C^\infty(\Omega) \xrightarrow{\text{grad}} C^\infty(\Omega;\mathbb{R}^3) \xrightarrow{\text{curl}} C^\infty(\Omega;\mathbb{R}^3) \xrightarrow{\text{div}} C^\infty(\Omega) \longrightarrow 0.$$

$$d^0 := \text{grad}, \quad d^1 := \text{curl}, \quad d^2 := \text{div}\,.$$

- complex property: $d^k \circ d^{k-1} = 0, \ \Rightarrow \ \mathcal{R}(d^{k-1}) \subset \mathcal{N}(d^k)$,
  $\text{curl} \circ \text{grad} = 0 \Rightarrow \mathcal{R}(\text{grad}) \subset \mathcal{N}(\text{curl}), \quad \text{div} \circ \text{curl} = 0 \Rightarrow \mathcal{R}(\text{curl}) \subset \mathcal{N}(\text{div})$
- cohomology: $\mathscr{H}^k := \mathcal{N}(d^k)/\mathcal{R}(d^{k-1})$,
  $\mathscr{H}^0 := \mathcal{N}(\text{grad}), \quad \mathscr{H}^1 := \mathcal{N}(\text{curl})/\mathcal{R}(\text{grad}), \quad \mathscr{H}^2 := \mathcal{N}(\text{div})/\mathcal{R}(\text{curl})$
- exactness (contractible domains): $\mathcal{N}(d^k) = \mathcal{R}(d^{k-1})$, i.e., $d^k u = 0 \ \Rightarrow \ u = d^{k-1} v$
  $\text{curl}\, u = 0 \Rightarrow u = \text{grad}\, \phi, \quad \text{div}\, v = 0 \Rightarrow v = \text{curl}\, \psi.$

Canonical finite elements for the de Rham complex



Raviart-Thomas (1977), Nédélec (1980) in numerical analysis

"The main advantage of these finite elements is the possibility of approximating Maxwell's equations while exactly verifying one of the physical law." – J.C. Nédélec, Mixed Finitc Elements in $\mathbb{R}^3$ (1980)

Bossavit (1988): differential forms and complex

"A rationale for the use of Lhese special 'mixed' clements can be obtained if one expresses basic equations in terms of differential forms, instead of vector fields. ... Whitney forms were described in 1957, long before the use of finite elements."
– A. Bossavit, Whitney forms: a class of finite elements for three-dimensional computations in electromagnetism (1988)

Hiptmair (1999), Arnold, Falk, Winther (2006): systematic study, "Finite Element Exterior Calculus"

Finite element exterior calculus (FEEC): structure-preserving FEM
Discrete exterior calculus (DEC): defining spaces and operators on primal and dual meshes
Topological data analysis (TDA): cohomology and Hodge-Laplacian on graphs

Lim, Lek-Heng. "Hodge Laplacians on graphs." SIAM Review 62.3 (2020).

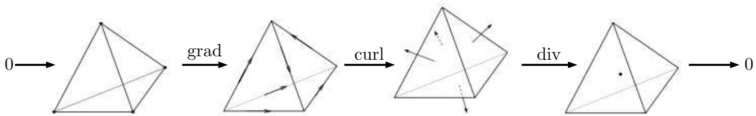
Why complexes matter?

Example: Gauss law in Maxwell equations. $\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E} = 0 \Longrightarrow \partial_t (\nabla \cdot \boldsymbol{B}) = 0$.
Typical Galerkin schemes: Find $\boldsymbol{E}_h \in Y_h, \ \boldsymbol{B}_h \in Z_h$, s.t.

$$(\partial_t \boldsymbol{B}_h, \boldsymbol{C}_h) + (\nabla \times \boldsymbol{E}_h, \boldsymbol{C}_h) = 0, \quad \forall \boldsymbol{C}_h \in Z_h.$$

Consequence: $\partial_t \boldsymbol{B}_h + \mathbb{P}\nabla \times \boldsymbol{E}_h = 0$, hence $\partial_t (\nabla \cdot \boldsymbol{B}_h) = -\nabla \cdot \mathbb{P}\nabla \times \boldsymbol{E}_h$. Non-zero in general, up to discretization errors.
$\mathbb{P}$: L^2 projection from $\nabla \times Y_h$ to Z_h. $\mathbb{P} = I$ (hence $\partial_t (\nabla \cdot \boldsymbol{B}_h) = 0$) only if $\nabla \times Y_h \subset Z_h$.

Relations like $\nabla \times Y_h \subset Z_h$ are encoded in differential complexes (homological algebra).



Using finite elements in complexes leads to (linear) constraint-preservation.

Discretization for MHD

Choice of spaces: all the spaces are in a finite element de Rham complex

- To preserve $\nabla \cdot \boldsymbol{B} = 0$, we discretize $\boldsymbol{B}$ in $H_0^h(\mathrm{div})$ and $\boldsymbol{E} \in H_0^h(\mathrm{curl})$.
  Complex: curl $H_0^h(\mathrm{curl}) \subset H_0^h(\mathrm{div})$!
- key cancellation for the magnetic helicity: on the continuous level,
$$\int \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) \cdot \boldsymbol{A} = \int (\boldsymbol{u} \times \boldsymbol{B}) \cdot \boldsymbol{B} = 0.$$
  On the discrete level, a natural mixed scheme yields
$$\int \nabla \times \mathbb{Q}_h^{\mathrm{curl}}(\boldsymbol{u} \times \boldsymbol{B}) \cdot \boldsymbol{A} = \int \mathbb{Q}_h^{\mathrm{curl}}(\boldsymbol{u} \times \boldsymbol{B}) \cdot \nabla \times \boldsymbol{A} = \int (\boldsymbol{u} \times \boldsymbol{B}) \cdot \mathbb{Q}_h^{\mathrm{curl}}\boldsymbol{B} \neq 0,$$
  where $\mathbb{Q}_h^{\mathrm{curl}}$ is the L^2 projection to $H_0^h(\mathrm{curl})$.
  Fix: introduce $\boldsymbol{H} = \mathbb{Q}_h^{\mathrm{curl}}\boldsymbol{B}$, use $\nabla \times \mathbb{Q}_h^{\mathrm{curl}}(\boldsymbol{u} \times \boldsymbol{H})$ in the scheme.
$$\int (\boldsymbol{u} \times \mathbb{Q}_h^{\mathrm{curl}}\boldsymbol{B}) \cdot \mathbb{Q}_h^{\mathrm{curl}}\boldsymbol{B} = 0.$$
- cross helicity: similar treatment as the magnetic helicity. Introduce $\boldsymbol{\omega} := \mathbb{Q}_h^{\mathrm{curl}}\nabla \times \boldsymbol{u}$ and modify the scheme accordingly: $\boldsymbol{\omega} \in H_h^0(\mathrm{curl})$ defined by
$$(\boldsymbol{\omega}, \boldsymbol{\mu}) - (\nabla \times \boldsymbol{u}, \boldsymbol{\mu}) = 0, \quad \forall \boldsymbol{\mu} \in H_h^0(\mathrm{curl}).$$

NUMERICAL SCHEME

Apply the same idea of choosing finite elements in a de Rham complex and adding projections :

Find $(\boldsymbol{B}, \boldsymbol{E}, \boldsymbol{H}, \boldsymbol{j}, \boldsymbol{u}) \in H^h(\mathrm{div}) \times H^h(\mathrm{curl}) \times H^h(\mathrm{curl}) \times H^h(\mathrm{curl}) \times H^h(\mathrm{div})$, such that for any $(\hat{\boldsymbol{B}}, \hat{\boldsymbol{E}}, \hat{\boldsymbol{H}}, \hat{\boldsymbol{j}}, \hat{\boldsymbol{u}})$ in the same space,

$$
\begin{aligned}
(\boldsymbol{B}_t, \hat{\boldsymbol{B}}) + (\nabla \times \boldsymbol{E}, \hat{\boldsymbol{B}}) &= 0, \\
(\boldsymbol{E}, \hat{\boldsymbol{E}}) &= -(\boldsymbol{u} \times \boldsymbol{H}, \hat{\boldsymbol{E}}), \\
(\boldsymbol{u}, \hat{\boldsymbol{v}}) &= \tau(\boldsymbol{j} \times \boldsymbol{H}, \hat{\boldsymbol{v}}), \\
(\boldsymbol{j}, \hat{\boldsymbol{j}}) &= (\boldsymbol{B}, \nabla \times \hat{\boldsymbol{j}}), \\
(\boldsymbol{H}, \hat{\boldsymbol{H}}) &= (\boldsymbol{B}, \hat{\boldsymbol{H}}).
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{B}_t + \nabla \times \boldsymbol{E} &= 0, \\
\boldsymbol{E} &= -\mathbb{P}(\boldsymbol{u} \times \boldsymbol{H}), \\
\boldsymbol{u} &= \tau\mathbb{Q}(\boldsymbol{j} \times \boldsymbol{H}), \\
\boldsymbol{j} &= \nabla_h \times \boldsymbol{B}, \\
\boldsymbol{H} &= \mathbb{P}\boldsymbol{B}.
\end{aligned}
$$

Energy law

$$
\frac{1}{2}\frac{d}{dt}\|\boldsymbol{B}\|^2 = -\tau\|\mathbb{Q}(\boldsymbol{H} \times \boldsymbol{j})\|^2.
$$

Helicity conservation

$$
\frac{d}{dt}\int \boldsymbol{A} \cdot \boldsymbol{B} = 0.
$$

Comparison scheme Find $(\boldsymbol{B}, \boldsymbol{u}) \in [H^h(\mathrm{grad})]^3 \times [H^h(\mathrm{grad})]^3$ such that for any $(\hat{\boldsymbol{B}}, \hat{\boldsymbol{u}})$ in the same space,

$$\begin{aligned}(\boldsymbol{B}_t, \hat{\boldsymbol{B}}) - (\boldsymbol{u} \times \boldsymbol{B}, \nabla \times \hat{\boldsymbol{B}}) &= 0, \\ (\boldsymbol{u}, \hat{\boldsymbol{u}}) &= \tau((\nabla \times \boldsymbol{B}) \times \boldsymbol{B}, \hat{\boldsymbol{B}}).\end{aligned}$$

Energy decays , but no helicity conservation and no (discrete) Hodge decomposition .

No mechanism preventing the numerical solution from decaying to zero (artificial magnetic reconnection can happen).

Problem 1: Hopf fibration

$$\boldsymbol{B}_0 = \frac{4\sqrt{a}}{\pi(1+r^2)^3}(2y(y-xz), -2(x+yz), (-1+x^2+y^2-z^2))$$

Every single field line of this field is a perfect circle, and every single field line is linked with every other one.

c.f. Smiet, C.B., Candelaresi, S. and Bouwmeester, D., 2017. Ideal relaxation of the Hopf fibration. Physics of Plasmas, 24(7).

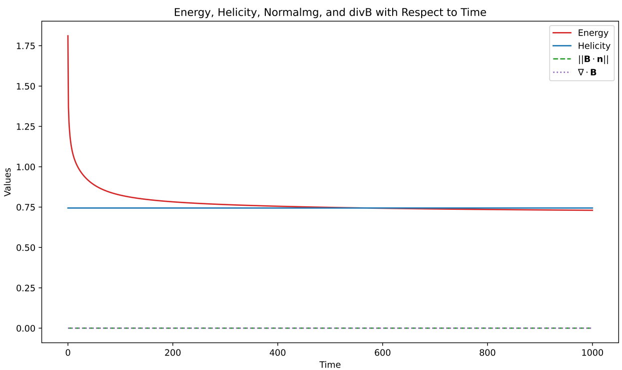


Figure. Helicity-preserving scheme

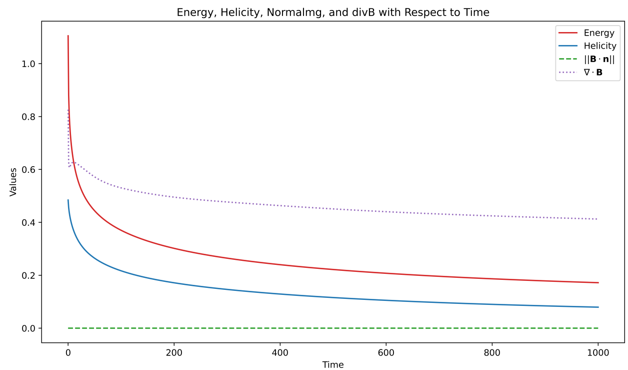
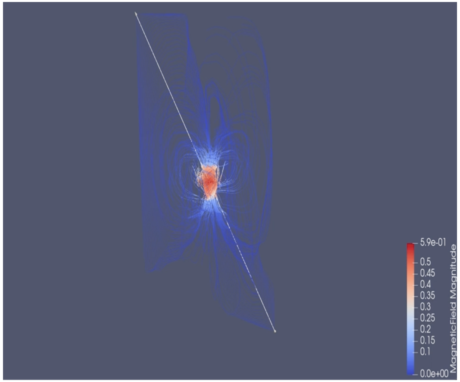
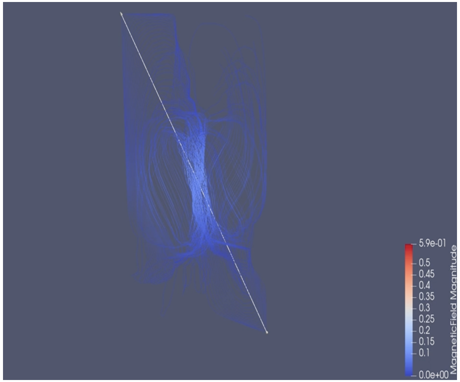
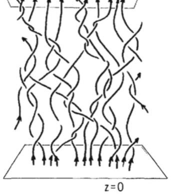


Figure. CG scheme (non-preserving)

$\tau = 10$, $dt = 1$ and $T = 1000$.


5.9e-01
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.0e+00
MagneticField Magnitude

5.9e-01
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.0e+00
MagneticField Magnitude

Problem 2: Parker problem



Additional difficulty: nontrivial cohomology (harmonic forms $\boldsymbol{B}_0 = (0,0,1)$) due to torus topology

$\nabla \times \boldsymbol{B}_0 = 0$, $\nabla \cdot \boldsymbol{B}_0 = 0$, $\boldsymbol{B}_0$ satisfies boundary conditions.

- Helicity not well defined.
- Harmonic part of $\boldsymbol{B}$ remains constant in time (because $\partial_t \boldsymbol{B} = -\nabla \times \boldsymbol{E}$ is in $\mathcal{R}(\mathrm{curl})$), but interferes the evolution of the exact ($\mathrm{curl}\, \boldsymbol{A}$) part.

Generalization of helicity:

- relative helicity: difficult to compute.
- A new definition: a combination of harmonic part and exact part

  $\boldsymbol{B} = \boldsymbol{B}_R + \boldsymbol{B}_H$; $\boldsymbol{B}_R = \mathrm{curl}\, \boldsymbol{A}$: exact part, $\boldsymbol{B}_H$: harmonic part.

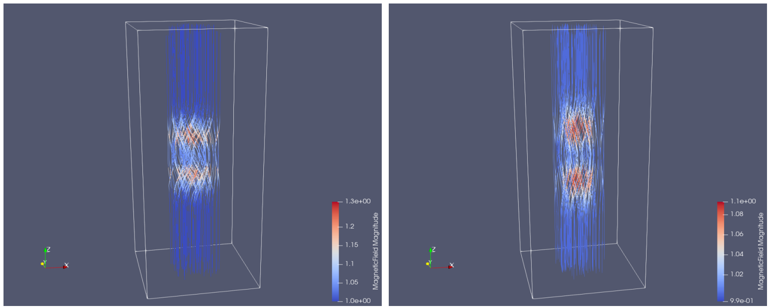
$$\tilde{H} := \int \boldsymbol{A} \cdot (\boldsymbol{B}_R + 2\boldsymbol{B}_H) = \int \boldsymbol{A} \cdot (\boldsymbol{B} + \boldsymbol{B}_H) \implies \frac{d}{dt}\tilde{H} = 0.$$

  Generalized Arnold inequality:

$$|\tilde{H}| \leq \|A\| \|B_R + 2B_H\| \leq C(\|B_R\|^2 + 4\|B_R + 2B_H\|^2).$$

Our algorithm: satisfies a discrete version of the above properties.

Initial results:



Left: initial state, Right: final state.

Extensions to full MHD (KH, Lee, Xu 2021)

Find $(\boldsymbol{u}, \omega, \boldsymbol{j}, \boldsymbol{E}, \boldsymbol{H}, \boldsymbol{B}, p) \in [H_0^h(\mathrm{curl}, \Omega)]^5 \times H_0^h(\mathrm{div}, \Omega) \times H_0^h(\mathrm{grad})$ such that

$$
\begin{aligned}
(D_t\boldsymbol{u}, \boldsymbol{v}) - (\boldsymbol{u} \times \omega, \boldsymbol{v}) + (\nabla p, \boldsymbol{v}) - S(\boldsymbol{j} \times \boldsymbol{H}, \boldsymbol{v}) &= (\boldsymbol{f}, \boldsymbol{v}), && \text{(1a)}\\
(\omega, \mu) - (\nabla \times \boldsymbol{u}, \mu) &= 0, && \text{(1b)}\\
(\boldsymbol{u}, \nabla q) &= 0, && \text{(1c)}\\
(D_t\boldsymbol{B}, \boldsymbol{C}) + (\nabla \times \boldsymbol{E}, \boldsymbol{C}) &= 0, && \text{(1d)}\\
(\boldsymbol{j}, \boldsymbol{k}) - (\boldsymbol{B}, \nabla \times \boldsymbol{k}) &= 0, && \text{(1e)}\\
(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{H}, \boldsymbol{G}) &= 0, && \text{(1f)}\\
(\boldsymbol{B}, \boldsymbol{F}) - (\boldsymbol{H}, \boldsymbol{F}) &= 0, && \text{(1g)}
\end{aligned}
$$

where $D_t\boldsymbol{u} = (\boldsymbol{u}^{new} - \boldsymbol{u}^{old})/\Delta t$, $D_t\boldsymbol{B} = (\boldsymbol{B}^{new} - \boldsymbol{B}^{old})/\Delta t$ and other variables are average of new and old values (*Crank-Nicolson time stepping*).

$$
\begin{aligned}
\boldsymbol{E} &= -\mathbb{Q}_h^{\mathrm{curl}}(\boldsymbol{u} \times \boldsymbol{H}),\\
\omega &= \mathbb{Q}_h^{\mathrm{curl}}(\nabla \times \boldsymbol{u})\\
\boldsymbol{j} &= \nabla_h \times \boldsymbol{B}, \quad \boldsymbol{H} = \mathbb{Q}_h^{\mathrm{curl}}\boldsymbol{B}.
\end{aligned}
$$

Structure-preserving properties hold.

Convergence test

h	$\|\boldsymbol{B} - \boldsymbol{B}_h\|_0$	order	$\|\boldsymbol{u} - \boldsymbol{u}_h\|_0$	order	$\|p - p_h\|_1$	order
2^{-2}	1.60E-3	x	4.15E-4	x	2.15E-4	x
2^{-3}	7.80E-4	1.04	2.18E-4	0.93	1.24E-4	0.79
2^{-4}	3.40E-4	1.20	1.05E-4	1.05	6.44E-5	0.95
2^{-5}	1.63E-4	1.06	5.30E-5	0.99	3.25E-5	0.99

Table. Convergence results for the MHD system. The error is computed at the time level $T = 1$ with the Crank-Nicolson time stepping with $\Delta t = 0.01$. $R_e = R_m = 10^4$.

CONVERGENCE

Algorithms converge well for *smooth true solutions*.

Theorem 1 (L. Beirão da Veiga, KH, L. Mascotto 2024[1])

Consider sequences $\{\mathcal{T}_h\}$ *of shape-regular, quasi-uniform meshes. Let the true solution be sufficiently smooth. Then, there exists a positive constant* C *independent of* h *such that, for all* t *in* $(0, T]$,

$$\|\mathbf{e}_h^{\mathbf{u}}(t)\|^2 + \|\mathbf{e}_h^{\mathbf{B}}(t)\|^2 + \int_0^t \|\operatorname{curl} \mathbf{e}_h^{\mathbf{u}}(s)\|^2 \, \mathrm{d}s + \int_0^t \|\mathbf{e}_h^{\mathbf{j}}(s)\|^2 \, \mathrm{d}s \leq C(\|\mathbf{e}_h^{\mathbf{u}}(0)\|^2 + \|\mathbf{e}_h^{\mathbf{B}}(0)\|^2 + h^{2(k+1)}).$$

The constant C *includes regularity terms of the numerical solution, the shape-regularity parameter of the mesh, and the polynomial degree* k.

Further question: What if the true solution is not smooth?

Onsager's conjecture concerns when energy conservation fails. For rough solutions, integration by parts fails and energy law does not hold any more. But most finite element methods are energy-conservative by construction. This means that structure-preserving methods must produce spurious solutions in such cases.

[1] L. Beirão da Veiga, KH, L. Mascotto, *Convergence analysis of a helicity-preserving finite element discretisation for an incompressible magnetohydrodynamics system*, arXiv (2024)

Summary:

Helicity characterizes fine topological structures. Breaking such fine structures leads to *wrong* qualitative behaviour in long-term evolution.

Ongoing and future questions:

- Solvers.
- Detecting discontinuities using *a posterior* estimators, numerical evidence (or counter-examples) of the Parker hypothesis.
- Convergence properties with rough solutions.

De Rham complex-based methods provide a possible computational approach for problems from Topological Hydrodynamics.

References

- *Knots in plasma*, CB Smiet, PhD thesis, Leiden University (2017).
  self-organization, Hopf fibration
- *The Parker problem: existence of smooth force-free fields and coronal heating.* D.I. Pontin, G. Hornig, Living Reviews in Solar Physics (2020). Parker problem
- *Helicity-conservative finite element discretization for incompressible MHD systems*, KH, Y.-J. Lee, J. Xu; Journal of Computational Physics (2021). Helicity-preserving schemes for MHD
- *Topology-preserving discretization for the magneto-frictional equations arising in the Parker conjecture*, M. He, P. E. Farrell, KH, B. Andrews, arXiv (2025). numerical methods
- *Convergence analysis of a helicity-preserving finite element discretisation for an incompressible magnetohydrodynamics system*, L. Beirão da Veiga, KH, L. Mascotto, arXiv (2024).
  convergence for smooth solutions