Modelling and computing generalized continua via complexes

– An example with linear Cosserat models–

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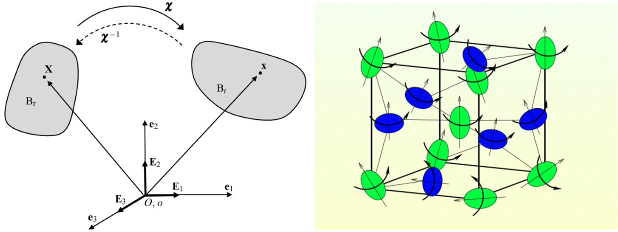
Outline

Cosserat elasticity (micropolar continuum): Cosserat brothers, Théorie des corps déformables (1909).

introduce a pointwise rotational degree of freedom, in addition to displacement in classical elasticity



Images: Left: José Merodio, Raymond Ogden, “Basic Equations of Continuum Mechanics”; Right: Elena F. Grekova, “Introduction to the mechanics of Cosserat media”

Related to Eringen: micropolar continua.

Motivation

Granular material

Images: Wikipedia

Even larger scales: ice floes (grains = icebergs), asteroid belts of the Solar System (grains = asteroids)...

Size effects Classical elasticity & plasticity: geometrically similar structures have same properties.
But realistic materials may not. Cosserat models incorporate grain sizes.

Cosserat models also inspired important mathematical developments, such as the concept of torsion.

Cartan's attempt at bridge-building between Einstein and the Cosserats – or how translational curvature became to be known as torsion. Scholz, E. E. EPJ H (2019).

Governing equations

Energy in the linear model: u: displacement (vector), ω: rotation (axial vector)

$$
\begin{aligned}
\mathcal{E}^{\mathrm{Cosserat}}(u,\omega) &:= \int_\Omega \Big(\frac{1}{2}\|\operatorname{grad} u - \operatorname{mskw}\omega\|^2_{C_1} + \frac{1}{2}\|\operatorname{grad}\omega\|^2_{C_2} - \langle f_u, u\rangle - \langle f_\omega,\omega\rangle\Big)\,dx \\
&= \int_\Omega \Big(\frac{1}{2}\|\operatorname{sym}\operatorname{grad} u\|^2_{\mathcal{C}} + \mu_c\,\|1/2\operatorname{curl} u - \omega\|^2 + \frac{\gamma+\beta}{2}\|\operatorname{sym}\operatorname{grad}\omega\|^2 \\
&\qquad + \frac{\gamma-\beta}{4}\|\operatorname{curl}\omega\|^2 + \frac{\alpha}{2}\|\operatorname{div}\omega\|^2\Big)\,dx - \int_\Omega \langle f_u,u\rangle + \langle f_\omega,\omega\rangle\,dx,
\end{aligned}
$$

with

$$
\begin{aligned}
C_1(\varepsilon) &= 2\mu\operatorname{sym}\varepsilon + \lambda\operatorname{tr}\varepsilon\, I + \mu_c\operatorname{skw}\varepsilon = \mathcal{C}(\varepsilon) + \mu_c\operatorname{skw}\varepsilon, \qquad \mathcal{C}(\varepsilon) = 2\mu\operatorname{sym}\varepsilon + \lambda\operatorname{tr}\varepsilon\, I, \\
C_2(\varepsilon) &= (\gamma+\beta)\operatorname{sym}\varepsilon + \alpha\operatorname{tr}\varepsilon\, I + (\gamma-\beta)\operatorname{skw}\varepsilon \\
&= (\gamma+\beta)\operatorname{dev}\operatorname{sym}\varepsilon + \frac{3\alpha+\beta+\gamma}{3}\operatorname{tr}\varepsilon\, I + (\gamma-\beta)\operatorname{skw}\varepsilon,
\end{aligned}
$$

where $\mathcal{C}$ is the classical elasticity tensor with *Lamé parameters* μ and λ, μ_c is the *Cosserat coupling* constant, and α,β,γ are additional *micropolar moduli*.

The additive coupling term $\operatorname{grad} u - \operatorname{mskw} w$ comes from linearization of a (multiplicative) action of Lie group element $\exp(\operatorname{mskw} w) \in \mathrm{SO}(3)$ on deformation φ.

Open: numerical methods robust with all the parameters.

WEAK AND STRONG COUPLING

A closer look at the energy:

$$\mathcal{E}^{\text{Cosserat}}(u,\omega) := \int_\Omega \left(\frac{1}{2} \| \operatorname{sym} \operatorname{grad} u \|^2_{\mathcal{C}} + \mu_c \| 1/2 \operatorname{curl} u - \omega \|^2 + \frac{1}{2} \| \operatorname{grad} \omega \|^2_{C_2} - \langle f_u, u \rangle - \langle f_\omega, \omega \rangle \right) dx$$

- $\mu_c = 0$: u and w decoupled. Solve a standard elasticity problem for u.
- $\mu_c = \infty$: "perfect coupling" - forcing $\omega = \frac{1}{2} \operatorname{curl} u$. The leading term becomes $\| \operatorname{grad} \omega \|^2_{C_2} = \frac{1}{2} \| \operatorname{grad} \operatorname{curl} u \|^2_{C_2}$. Mixed 4th-2nd order problems: couple stress model.

  So parameter-robust method for Cosserat should also solve couple stress models.

Existing work: *Mixed finite element methods for linear Cosserat equations*. Boon, W. M., Duran, O., & Nordbotten, J. M., arXiv preprint (2024).

$\mu_c > 0$, $\mu_c \to 0$.

de Rham complex (3D version)

$$0 \longrightarrow C^\infty(\Omega) \xrightarrow{\text{grad}} C^\infty(\Omega;\mathbb{R}^3) \xrightarrow{\text{curl}} C^\infty(\Omega;\mathbb{R}^3) \xrightarrow{\text{div}} C^\infty(\Omega) \longrightarrow 0.$$

$$d^0 := \text{grad}, \quad d^1 := \text{curl}, \quad d^2 := \text{div}\,.$$

- complex property: $d^k \circ d^{k-1} = 0, \;\Rightarrow\; \mathcal{R}(d^{k-1}) \subset \mathcal{N}(d^k)$,
  $\text{curl} \circ \text{grad} = 0 \Rightarrow \mathcal{R}(\text{grad}) \subset \mathcal{N}(\text{curl}), \quad \text{div} \circ \text{curl} = 0 \Rightarrow \mathcal{R}(\text{curl}) \subset \mathcal{N}(\text{div})$
- cohomology: $\mathscr{H}^k := \mathcal{N}(d^k)/\mathcal{R}(d^{k-1})$,
  $\mathscr{H}^0 := \mathcal{N}(\text{grad}), \quad \mathscr{H}^1 := \mathcal{N}(\text{curl})/\mathcal{R}(\text{grad}), \quad \mathscr{H}^2 := \mathcal{N}(\text{div})/\mathcal{R}(\text{curl})$
- exactness (contractible domains): $\mathcal{N}(d^k) = \mathcal{R}(d^{k-1})$, i.e., $d^k u = 0 \;\Rightarrow\; u = d^{k-1} v$
  $\text{curl}\, u = 0 \Rightarrow u = \text{grad}\,\phi, \quad \text{div}\, v = 0 \Rightarrow v = \text{curl}\,\psi.$

In higher dimensions,

$$\cdots \longrightarrow \Lambda^{k-1} \xrightarrow{d^{k-1}} \Lambda^k \xrightarrow{d^k} \Lambda^{k+1} \longrightarrow \cdots$$

Λ^k : differential k-forms, $\quad d^k$: exterior derivatives

A DIFFERENTIAL COMPLEX POINT OF VIEW

From complexes to PDEs

Formal adjoint of operators:

$$\operatorname{grad}^* = -\operatorname{div}, \quad \operatorname{curl}^* = \operatorname{curl}, \quad \operatorname{div}^* = -\operatorname{grad}.$$

$\int_\Omega \operatorname{grad} u \cdot v = -\int_\Omega u \operatorname{div} v + \text{bound. term}, \quad \int_\Omega \operatorname{curl} u \cdot v = \int_\Omega u \cdot \operatorname{curl} v + \text{bound. term}$

$$(\operatorname{grad} u, v) = (u, -\operatorname{div} v), \qquad (\operatorname{curl} u, v) = (u, \operatorname{curl} v)$$

Formal adjoint of de Rham complex:

$$0 \longleftarrow C^\infty(\Omega) \xleftarrow{-\operatorname{div}} C^\infty(\Omega;\mathbb{R}^3) \xleftarrow{\operatorname{curl}} C^\infty(\Omega;\mathbb{R}^3) \xleftarrow{-\operatorname{grad}} C^\infty(\Omega) \longleftarrow 0.$$

$$d_2^* := -\operatorname{div}, \quad d_1^* := \operatorname{curl}, \quad d_0^* := -\operatorname{grad}.$$

A DIFFERENTIAL COMPLEX POINT OF VIEW

connections to PDEs: Hodge-Laplacian problems.

$$(d^{k-1}d_{k-1}^{*} + d_k^{*}d^k)u = f.$$

connections to PDEs: Hodge-Laplacian problems.

$$(d^{k-1}d^*_{k-1} + d^*_k d^k)u = f.$$

$$0 \rightleftarrows C^\infty(\Omega) \underset{-\operatorname{div}}{\overset{\operatorname{grad}}{\rightleftarrows}} C^\infty(\Omega;\mathbb{R}^3) \qquad C^\infty(\Omega;\mathbb{R}^3) \qquad C^\infty(\Omega) \qquad 0.$$

Hodge-Laplacian problem:

$$-\operatorname{div}\operatorname{grad} u = f.$$

Poisson equation.

Variational form (energy):

$$\inf_u \frac{1}{2}\|\nabla u\|^2 - \int_\Omega fu.$$

A DIFFERENTIAL COMPLEX POINT OF VIEW

connections to PDEs: Hodge-Laplacian problems.

$$(d^{k-1}d^*_{k-1} + d^*_k d^k)u = f.$$

$$0 \qquad C^\infty(\Omega) \underset{-\operatorname{div}}{\overset{\operatorname{grad}}{\rightleftarrows}} C^\infty(\Omega;\mathbb{R}^3) \underset{\operatorname{curl}}{\overset{\operatorname{curl}}{\rightleftarrows}} C^\infty(\Omega;\mathbb{R}^3) \qquad C^\infty(\Omega) \qquad 0.$$

Hodge-Laplacian problem:

$$-\operatorname{grad}\operatorname{div} v + \operatorname{curl}\operatorname{curl} v = f.$$

Maxwell equations.

Variational form (energy):

$$\inf_v \frac{1}{2}(\|\operatorname{curl} v\|^2 + \|\operatorname{div} v\|^2) - \int_\Omega fv.$$

A DIFFERENTIAL COMPLEX POINT OF VIEW

connections to PDEs: Hodge-Laplacian problems.

$$(d^{k-1}d^*_{k-1} + d^*_k d^k)u = f.$$

$$0 \qquad C^\infty(\Omega) \qquad C^\infty(\Omega;\mathbb{R}^3) \underset{\text{curl}}{\overset{\text{curl}}{\rightleftarrows}} C^\infty(\Omega;\mathbb{R}^3) \underset{-\text{grad}}{\overset{\text{div}}{\rightleftarrows}} C^\infty(\Omega) \qquad 0.$$

Hodge-Laplacian problem:

$$\text{curl}\,\text{curl}\, v - \text{grad}\,\text{div}\, v = f.$$

Maxwell equations.

Variational form (energy):

$$\inf_v \frac{1}{2}(\|\text{curl}\, v\|^2 + \|\text{div}\, v\|^2) - \int_\Omega fv.$$

A DIFFERENTIAL COMPLEX POINT OF VIEW

connections to PDEs: Hodge-Laplacian problems.

$$(d^{k-1} d^*_{k-1} + d^*_k d^k) u = f.$$

$$0 \qquad C^\infty(\Omega) \qquad C^\infty(\Omega; \mathbb{R}^3) \qquad C^\infty(\Omega; \mathbb{R}^3) \underset{-\operatorname{grad}}{\overset{\operatorname{div}}{\rightleftarrows}} C^\infty(\Omega) \rightleftarrows 0.$$

Hodge-Laplacian problem:

$$-\operatorname{div} \operatorname{grad} u = f.$$

Poisson equation.

Variational form (energy):

$$\inf_u \frac{1}{2} \|\nabla u\|^2 - \int_\Omega f u.$$

How to derive more complexes: the BGG machinery

Bernstein-Gelfand-Gelfand (BGG) machinery: Derive complexes from de Rham complexes; carry over de Rham results. (B-G-G 1975, Čap,Slovák,Souček 2001, Eastwood 2000, Arnold,Falk,Winther 2006)

BGG diagram: complexes connected by algebraic operators in a (anti)commuting diagram ($dS = -Sd$)

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & V^{k-2} & \xrightarrow{d^{k-2}} & V^{k-1} & \xrightarrow{d^{k-1}} & V^{k} & \xrightarrow{d^{k}} & V^{k+1} & \longrightarrow \cdots \\
 & & \nearrow^{S^{k-2}} & & \nearrow^{S^{k-1}} & & \nearrow^{S^{k}} & & \\
\cdots \longrightarrow & W^{k-2} & \xrightarrow{d^{k-2}} & W^{k-1} & \xrightarrow{d^{k-1}} & W^{k} & \xrightarrow{\mathrm{div}} & W^{k+1} & \longrightarrow \cdots
\end{array}$$

Two complexes can be derived from the above BGG diagram:
twisted complex:

$$\cdots \longrightarrow \begin{pmatrix} V^{k-1} \\ W^{k-1} \end{pmatrix} \xrightarrow{\begin{pmatrix} d^{k-1} & -S^{k-1} \\ 0 & d^{k-1} \end{pmatrix}} \begin{pmatrix} V^{k} \\ W^{k} \end{pmatrix} \xrightarrow{\begin{pmatrix} d^{k} & -S^{k} \\ 0 & d^{k} \end{pmatrix}} \begin{pmatrix} V^{k+1} \\ W^{k+1} \end{pmatrix} \longrightarrow \cdots$$

BGG diagram: eliminating components connected by $S^{\bullet}$

BGG diagram in 1D:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & H^2 & \xrightarrow{\partial_x} & H^1 & \longrightarrow & 0 \\
 & & & \overset{I}{\nearrow} & & & \\
0 & \longrightarrow & H^1 & \xrightarrow{\partial_x} & L^2 & \longrightarrow & 0.
\end{array}
$$

Twisted complex:

$$
0 \longrightarrow \begin{pmatrix} H^1 \\ H^1 \end{pmatrix} \xrightarrow{\begin{pmatrix} \frac{d}{dx} & -I \\ 0 & \frac{d}{dx} \end{pmatrix}} \begin{pmatrix} L^2 \\ L^2 \end{pmatrix} \longrightarrow 0.
$$

Energy of Hodge-Laplacian:

$$
\|\frac{d}{dx}w - \varphi\|^2_{C_1} + \|\frac{d}{dx}\varphi\|^2_{C_2}
$$

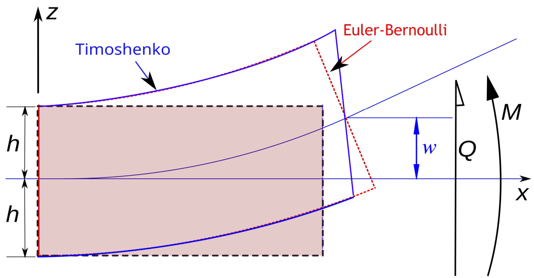
.

BGG complex:

$$
0 \longrightarrow H^2 \xrightarrow{\partial_x^2} L^2 \longrightarrow 0.
$$

Energy of Hodge-Laplacian

$$
\|\frac{d^2}{dx^2}w\|^2_C.
$$



Images: Wikipedia

3D elasticity: elasticity (Kröner, Calabi) complex

displacement formulation

Kröner's continuum description of dislocations/defects, internal stress

$$0 \longrightarrow C^\infty \otimes \mathbb{V} \xrightarrow{\text{def}} C^\infty \otimes \mathbb{S} \xrightarrow{\text{inc}} C^\infty \otimes \mathbb{S} \xrightarrow{\text{div}} C^\infty \otimes \mathbb{V} \longrightarrow 0$$

displacement *strain (metric)* *stress (curvature)* *force*

intrinsic elasticity (Ciarlet et al.)

Hellinger-Reissner principle of elasticity

$\mathbb{V} := \mathbb{R}^3$ vectors, $\mathbb{S} := \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ symmetric matrices

$$\operatorname{def} u := 1/2(\nabla u + \nabla u^T), \quad (\operatorname{def} u)_{ij} = 1/2(\partial_i u_j + \partial_j u_i).$$

$$\operatorname{inc} g := \nabla \times g \times \nabla, \quad (\operatorname{inc} g)^{ij} = \epsilon^{ikl}\epsilon^{jst}\partial_k\partial_s g_{lt}.$$

$$\operatorname{div} v := \nabla \cdot v, \quad (\operatorname{div} v)_i = \partial^j u_{ij}.$$

g metric $\Rightarrow$ $\operatorname{inc} g$ linearized Einstein tensor ($\backsimeq \mathrm{Riem} \backsimeq \mathrm{Ric}$ in 3D)

$\operatorname{inc} \circ \operatorname{def} = 0$: Saint-Venant compatibility

$\operatorname{div} \circ \operatorname{inc} = 0$: Bianchi identity

SKETCH OF DERIVATION: COMPLEXES FROM COMPLEXES

Step 1: connect two (or more) de Rham complexes

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathbb{R}^3 & \xrightarrow{\mathrm{grad}} & \mathbb{R}^{3\times 3} & \xrightarrow{\mathrm{curl}} & \mathbb{R}^{3\times 3} & \xrightarrow{\mathrm{div}} & \mathbb{R}^3 & \longrightarrow & 0\\
 & & & {\scriptstyle -\,\mathrm{mskw}}\nearrow & & {\scriptstyle \mathcal{S}}\nearrow & & {\scriptstyle 2\,\mathrm{vskw}}\nearrow & & & \\
0 & \longrightarrow & \mathbb{R}^3 & \xrightarrow{\mathrm{grad}} & \mathbb{R}^{3\times 3} & \xrightarrow{\mathrm{curl}} & \mathbb{R}^{3\times 3} & \xrightarrow{\mathrm{div}} & \mathbb{R}^3 & \longrightarrow & 0
\end{array}$$

$\mathcal{S}u := u^T - \mathrm{tr}(u)I$, bijective

Step 2: eliminate as much as possible

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathbb{R}^3 & \xrightarrow{\mathrm{grad}} & \mathbb{S}+\mathbb{K} & \xrightarrow{\mathrm{curl}} & \mathbb{R}^{3\times 3} & \xrightarrow{\mathrm{div}} & \mathbb{R}^3 & \longrightarrow & 0\\
 & & & {\scriptstyle -\,\mathrm{mskw}}\nearrow & & {\scriptstyle \mathcal{S}}\nearrow & & {\scriptstyle 2\,\mathrm{vskw}}\nearrow & & & \\
0 & \longrightarrow & \mathbb{R}^3 & \xrightarrow{\mathrm{grad}} & \mathbb{R}^{3\times 3} & \xrightarrow{\mathrm{curl}} & \mathbb{S}+\mathbb{K} & \xrightarrow{\mathrm{div}} & \mathbb{R}^3 & \longrightarrow & 0
\end{array}$$

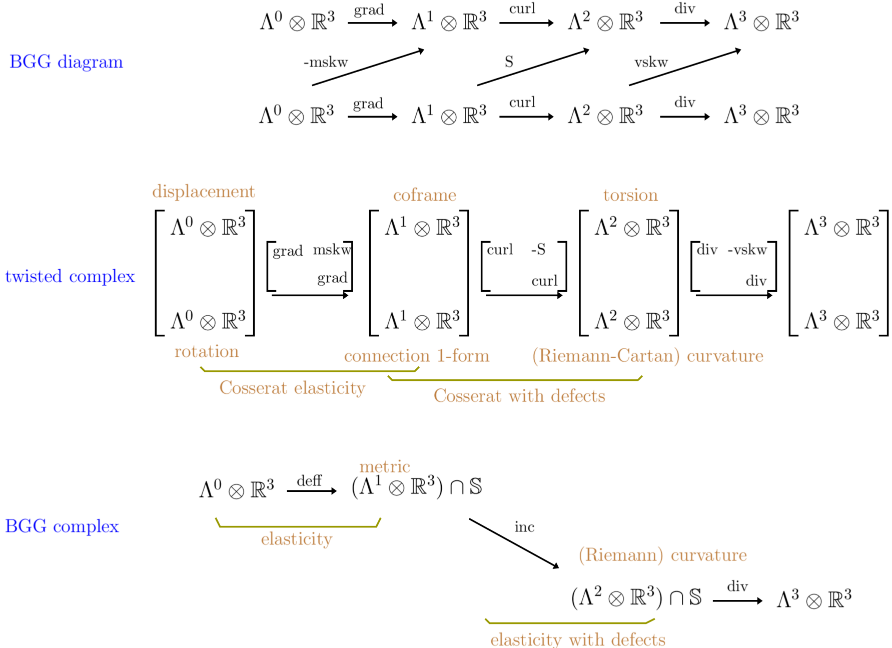
$\mathbb{S}$: symmetric matrix, $\mathbb{K}$: skew-symmetric matrix

Step 3: connect rows by zig-zag

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{R}^3 & \xrightarrow{\mathrm{sym\,grad}} & \mathbb{S} & \xrightarrow{\mathrm{curl}} & & & \\
 & & & & & \swarrow & & & \\
 & & & & & \xrightarrow{\mathrm{curl}^{\mathrm{T}}} \mathbb{S} & \xrightarrow{\mathrm{div}} & \mathbb{R}^3 & \longrightarrow 0.
\end{array}$$

Conclusion: the cohomology of the output (elasticity) is isomorphic to the input (de Rham)

A CLOSER LOOK AT THE DERIVATION: TWISTED COMPLEXES



In 1D, 2D, 3D:

- twisted complexes: Timoshenko beam, Reissner-Mindlin plate, Cosserat elasticity
- BGG complexes: Euler-Bernoulli beam, Kirchhoff-Love plate, standard elasticity.

Mechanics interpretation of BGG construction: eliminating microstructure variables (e.g., pointwise rotation) or torsion from twisted complexes via cohomology-preserving projections.

Part of a larger picture... mechanics v.s. complexes v.s. geometry

Trace complexes: dimension reduction

$$
\begin{array}{ccc}
\text{Cosserat (micropolar) elasticity} & \xrightarrow{\text{BGG (elasticity)}} & \text{classical elasticity} \\
\Big\downarrow \Gamma\text{-convergence} & & \Big\downarrow \Gamma\text{-convergence} \\
\text{Reissner} - \text{Mindlin plate} & \xrightarrow{\text{BGG (hessian)}} & \text{Kirchhoff plate}
\end{array}
$$

Γ convergence: *The Reissner–Mindlin plate is the Γ-limit of Cosserat elasticity*. Neff, P., Hong, K. I., & Jeong, J. M3AS, (2010).

High order forms: continuum defect theory

Idea (Kröner, Nye etc.): strain in standard elasticity $e = \operatorname{sym}\operatorname{grad}(u)$ satisfying $\operatorname{inc} e = 0$ (Saint-Venant compatibility). Defects lead to incompatibility: use e as a basic variable, and in general $\operatorname{inc} e \neq 0$ describes defects.

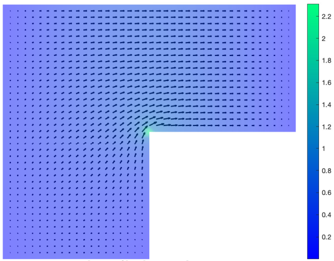
Other types of microstructures (dilation? rotation+dilation? Lie groups?), nonlinear and curved (shell) theories etc. Towards an "*Erlangen program for generalized continuum*".

Finite element methods

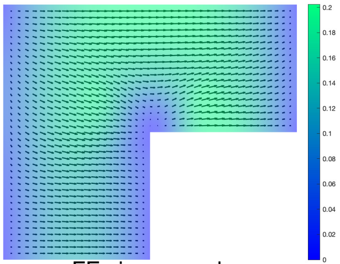
USE FINITE ELEMENTS FROM A COMPLEX...

A model problem for couple stress (continua with microstructures)

$$-\operatorname{curl}\Delta\operatorname{rot}\boldsymbol{u}-\operatorname{grad}\operatorname{div}\boldsymbol{u}=\boldsymbol{f}$$



scalar finite elements



FEs in a complex

model problem for generalised continua, classical finite element goes wrong.

KH,Q.Zhang,J.Han,L.Wang,Z.Zhang (2022) *Spurious solutions for high order curl problems*, IMA.

More examples in FEEC book/papers.

Things work when the we discrete the entire complex and preserve the cohomology.

Generalizing finite elements: back to de Rham's currents

Obtaining parameter-robust schemes for Cosserat: discretizing the entire BGG diagram. However, conforming discretization requires redundant d.o.f.s and may not be robust with thickness.

A more canonical discretization: use currents (measures, Dirac delta), rather than functions.

Geometric Measure Theory , graphics

(Codimensional geometry: A point cloud represents a probability measure; curve cloud, surface cloud...)

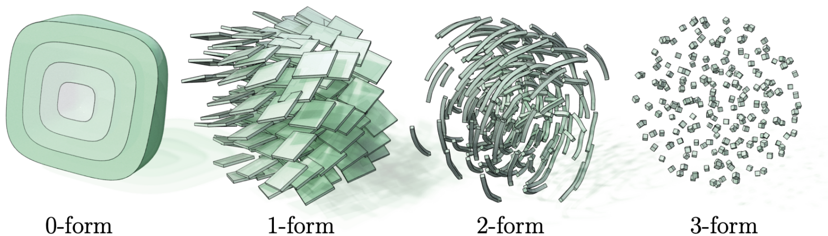
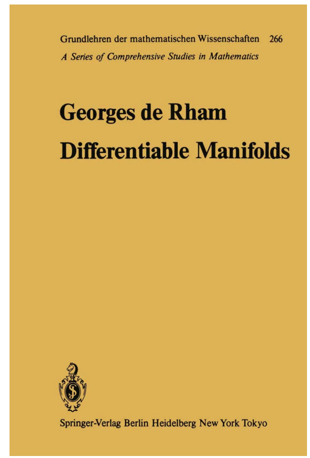


Figure 2.4 Differential k-forms can be represented by clouds of codimension-k geometries.

Figure: *Exterior Calculus in Graphics*, Stephanie Wang, Mohammad Sina Nabizadeh and Albert Chern; ACM SIGGRAPH 2023 courses.



Solving PDEs using distributional elements

General principle: evaluating Dirac delta only on continuous functions.

Poisson (trivial example):

$$\int \nabla u \cdot \nabla v = \int fv, \ \forall v \in \text{Lagrange}.$$

What if we view it as $\langle -\Delta u, v\rangle = (f, v)$?

- $u \in C^0$, $\mathrm{grad}\, u \in$ Nédélec (normal components may not be continuous).
- div (in the sense of distributions): $\mathrm{grad}\, u \mapsto$ Dirac delta on faces.
- $\Delta u = \mathrm{div}\, \mathrm{grad}\, u$ (as a delta) can be paired with v (single-valued on faces!).

Elasticity and TDNNS

$$-\operatorname{div} C^{-1} \operatorname{sym} \operatorname{grad} u = f.$$

Weak form: find $\sigma \in \Sigma_h$, $u \in V_h$, such that

$$\begin{aligned} (\sigma, \tau)_C + (\operatorname{div} \tau, u) &= 0, \quad \forall \tau, \\ (\operatorname{div} \sigma, v) &= -(f, v), \quad \forall v. \end{aligned}$$

u: displacement (vector); σ: stress (symmetric matrix)

Question: how to choose Σ_h and V_h (such that the pair $(\operatorname{div} \sigma, v) = \int_\Omega \operatorname{div} \sigma \cdot v$ satisfies inf-sup condition?

- displacement formulation: $u \in C^0$, $\sigma \in C^{-1}$, $-\int_\Omega \tau : \operatorname{sym} \operatorname{grad}(u)$. locking
- Hellinger-Reissner principle: $u \in C^{-1}$, $\sigma \in C^n$ ($\sigma \cdot n$ continuous), $\int_\Omega \operatorname{div} \tau \cdot u$. difficult to construct
- TDNNS (Pechstein, Schöberl 2011): $u \in C^t$ ($u \cdot t$ continuous), $\sigma \in C^{nn}$ ($n \cdot \sigma \cdot n$ continuous)

  $\operatorname{div} \sigma = \sum_{F \in \mathscr{F}} [\sigma]_{tn} \delta_F$: tangential delta, $\langle \operatorname{div} \sigma, v \rangle = \sum_{F \in \mathscr{F}} \int_F [\sigma]_{tn} \cdot v$ well defined.

  Robust with thickness/anisotropy (3D TDNNS restricted to face is a 2D TDNNS).

Tangential-displacement and normal–normal-stress continuous mixed finite elements for elasticity. Pechstein, A., & Schöberl, J., M3AS (2011)

Stokes and MCS

Introduce a stress-like variable $\sigma = \nabla u$ (trace-free):

$$\begin{aligned} -\operatorname{div}\sigma + \nabla p &= f, \\ \sigma &= \nabla u, \\ \nabla \cdot u &= 0. \end{aligned}$$

Weak form: $(\sigma, \tau) - (\tau, \nabla u) = 0$. Motivation: using $H(\mathrm{div})$ element to discretize u ($u \cdot n$ continuous).

Then $\nabla u = \sum_{F \in \mathscr{F}} [u_t] \otimes n \delta_F$. Pair $\langle \nabla u, \tau \rangle = \sum_{F \in \mathscr{F}} \int_F ([u_t] \otimes n) : \tau$ well defined if $t \cdot \tau \cdot n$ is continuous.

τ : piecewise constant trace-free matrix, $n \cdot \tau \cdot n$ as d.o.f.s.

A mass conserving mixed stress formulation for the Stokes equations. Gopalakrishnan, J., Lederer, P. L., & Schöberl, J., IMA (2020)

BACK TO COSSERAT: TWO SCHEMES

Idea of Scheme 1 (MCS):

$$\begin{array}{ccc} u & \xrightarrow{\text{grad}} & \\ & {\scriptstyle -\,\text{mskw}} \nearrow & \\ \omega & \xrightarrow{\text{grad}} & m', \end{array}$$

u : Lagrange (displacement formulation). To avoid coupling locking (robustness with μ_c):

> space of ω should be large enough to contain $-\,\text{vskw} \circ \text{grad}\, u = -\,\text{curl}\, u \quad \Longrightarrow \quad$ discretize ω in RT.

Then $\text{grad}\,\omega$ is a distribution (the MCS situation!). We introduce m (trace-free, *tn*-continous) to accommodate $\text{grad}\,\omega$.

Idea of Scheme 2 (MCS-TDNNS): Displacement formulation still suffers from volume locking (Lamé const $\to \frac{1}{2}$) as in standard elasticity. Further introduce TDNNS idea to fix this.

$$\begin{array}{ccc} u & \xrightarrow{\text{grad}} & \sigma' \\ & {\scriptstyle -\,\text{mskw}} \nearrow & \\ \omega & \xrightarrow{\text{grad}} & m'. \end{array}$$

$u \in$ Nédélec, introducing σ with *nn* continuous to accommodate $\text{grad}\, u$.

Problem 1 (MCS and MCS-TDNNS mixed methods for linear Cosserat elasticity)

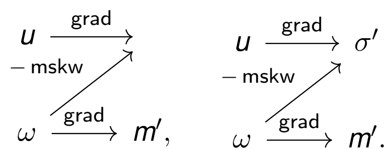
Find $(u, \omega, m) \in [\mathrm{Lag}^k]^3 \times \mathrm{RT}^{k-1} \times \mathrm{MCS}^{k-1}$ *solving the Lagrangian*

$$\mathcal{L}^m(u, \omega, m) = \frac{1}{2}\int_\Omega \left(\|\operatorname{grad} u - \operatorname{mskw}\omega\|^2_{C_1} - \|m\|^2_{C_2^{-1}}\right)\,dx - \langle \operatorname{div} m, \omega\rangle_{H_0(\operatorname{div})^*} - f(u, \omega, m) \to \min_{u,\omega}\max_{m}.$$

Find $(u, \omega, m, \sigma) \in \mathrm{Ned}^k_{II} \times \mathrm{RT}^{k-1} \times \mathrm{MCS}^{k-1} \times \mathrm{HHJ}^k$ *solving the Lagrangian*

$$\mathcal{L}^{m,\sigma}(u, \omega, \sigma, m) = \frac{1}{2}\int_\Omega \left(-\|\sigma\|^2_{\mathbb{C}^{-1}} + 2\mu_c\,\|1/2\operatorname{curl} u - \omega\|^2 - \|m\|^2_{C_2^{-1}}\right)\,dx - \langle \operatorname{div}\sigma, u\rangle_{H_0(\operatorname{curl})^*}$$
$$- \langle \operatorname{div} m, \omega\rangle_{H_0(\operatorname{div})^*} - f(u, \omega, m, \sigma) \to \min_{u,\omega}\max_{m,\sigma}.$$

The MCS and MCS-TDNNS formulations are based on the following diagram:



$$\mathrm{MCS}^k := \{\sigma_h \in [\mathcal{P}^k(\mathcal{T})]^{3\times 3} \;:\; \langle n_F \times \sigma_h, n_F\rangle \text{ is continuous across all faces } F \in \mathcal{F}\}.$$

$$\mathrm{HHJ}^k := \{\sigma_h \in [\mathcal{P}^k(\mathcal{T})]^{3\times 3}_{\mathrm{sym}} \;:\; \sigma_{h,n_F n_F} := \langle \sigma_h n_F, n_F\rangle \text{ is continuous across all faces } F \in \mathcal{F}\}.$$

Theorem 1

The mixed form is well-posed and there holds with $\gamma_h = 2\mu_c({}^1/_2 \operatorname{curl} u_h - \omega_h)$ *the following stability estimate*

$$\|m_h\|_{L^2} + \|\sigma_h\|_{L^2} + \|\gamma_h\|_\Gamma + \|u_h\|_{V_h} + \|\omega_h\|_{W_h} + \sqrt{\mu_c}\,\|{}^1/_2 \operatorname{curl} u_h - \omega_h\|_{L^2} \leq C\big(\|f_u\|_{L^2} + \|f_\omega\|_{L^2}\big),$$

where $C > 0$ *is a constant independent of* μ_c *and the norms* $\|\cdot\|_\Gamma$, $\|\cdot\|_{V_h}$, *and* $\|\cdot\|_{W_h}$ *are given by*

$$\|u\|^2_{V_h} = \sum_{T\in\mathcal{T}} \|\operatorname{sym}\operatorname{grad} u\|^2_{L^2(T)} + \frac{1}{h}\sum_{F\in\mathcal{F}} \|[\![u_n]\!]\|^2_{L^2(F)}, \qquad \|\gamma\|_\Gamma = \frac{1}{\sqrt{\mu_c}}\|\gamma\|_{L^2}$$

$$\|\omega\|^2_{W_h} = \sum_{T\in\mathcal{T}} \|\operatorname{grad}\omega\|^2_{L^2(T)} + \frac{1}{h}\sum_{F\in\mathcal{F}} \|[\![\omega_t]\!]\|^2_{L^2(F)}.$$

Proof: Use MCS and TDNNS inf-sup results. Track inf-sup constants with properly scaled norms - independent of μ_c.

Theorem 2 (Convergence)

Let (u,ω,m,σ) be the exact solution of linear Cosserat elasticity and $(u_h,\omega_h,m_h,\sigma_h,\gamma_h)\in \mathrm{Ned}_{II}^k\times \mathrm{RT}^{k-1}\times \mathrm{MCS}^{k-1}\times \mathrm{HHJ}^k\times \mathrm{RT}^{k-1}$ the discrete solution with homogeneous Dirichlet data on the whole boundary. Assume for $0\leq l\leq k-1$ the regularity $u\in[H^1(\Omega)]^3\cap[H^{l+1}(\mathcal{T})]^3$, $\omega\in[H^1(\Omega)]^3\cap[H^{l+1}(\mathcal{T})]^3$, $m\in[H^1(\Omega)]^3\cap[H^{l+1}(\mathcal{T})]^{3\times 3}$, and $\sigma\in[H^1(\Omega)]^3\cap[H^{l+1}(\mathcal{T})]^{3\times 3}$. Then there holds the convergence estimate

$$\begin{aligned}&\|u-u_h\|_{V_h}+\|\omega-\omega_h\|_{W_h}+\|m-m_h\|_{L^2}+\|\sigma-\sigma_h\|_{L^2}+\|\gamma-\gamma_h\|_\Gamma\\&\qquad\leq ch^l\big(\|u\|_{H^{l+1}(\Omega)}+\|\omega\|_{H^{l+1}(\Omega)}+\|m\|_{H^l(\Omega)}+\|\sigma\|_{H^l(\Omega)}+\frac{1}{\sqrt{\mu_c}}\|\gamma\|_{H^l(\Omega)}\big),\\&\|u-u_h\|_{V_h}+\|\omega_h-\mathcal{I}^{\mathrm{RT},k-1}\omega\|_{W_h}+\|m-m_h\|_{L^2}+\|\sigma-\sigma_h\|_{L^2}+\|\gamma-\gamma_h\|_\Gamma\\&\qquad\leq ch^{l+1}\big(\|u\|_{H^{l+2}(\Omega)}+\|m\|_{H^{l+1}(\Omega)}+\|\sigma\|_{H^{l+1}(\Omega)}+\frac{1}{\sqrt{\mu_c}}\|\gamma\|_{H^{l+1}(\Omega)}\big),\end{aligned}$$

where the discrete norms $\|\cdot\|_{V_h}$ and $\|\cdot\|_{W_h}$ are given by

$$\|u\|^2_{V_h}=\sum_{T\in\mathcal{T}}\|\operatorname{sym}\operatorname{grad}u\|^2_{L^2(T)}+\frac{1}{h}\sum_{F\in\mathcal{F}}\|[\![u_n]\!]\|^2_{L^2(F)},\quad \|\omega\|^2_{W_h}=\sum_{T\in\mathcal{T}}\|\operatorname{grad}\omega\|^2_{L^2(T)}+\frac{1}{h}\sum_{F\in\mathcal{F}}\|[\![\omega_t]\!]\|^2_{L^2(F)}.$$

Second estimate: superconvergence for ω.

MCS FOR COUPLE STRESS PROBLEM

Limit $\mu_c \to \infty$: couple stress problem.

Find $(u_h, m_h) \in [\mathrm{Lag}^k]^3 \times \mathrm{MCS}^{k-1}$ solving the Lagrangian (with appropriate boundary conditions)

$$\mathcal{L}_{\mathrm{MCS}}^{\mathrm{CoupleStress}}(u_h, m_h) = \frac{1}{2}\int_\Omega \left(\| \operatorname{sym} \operatorname{grad} u_h\|_{\mathbb{C}}^2 - \|m_h\|_{C_2^{-1}}^2 \right) dx - \frac{1}{2}\langle \operatorname{div} m_h, \operatorname{curl} u_h \rangle_{H(\operatorname{div})^*}$$
$$- f(u_h, m_h) \to \min_{u_h} \max_{m_h} .$$

Numerical tests: Cylindrical bending of plate

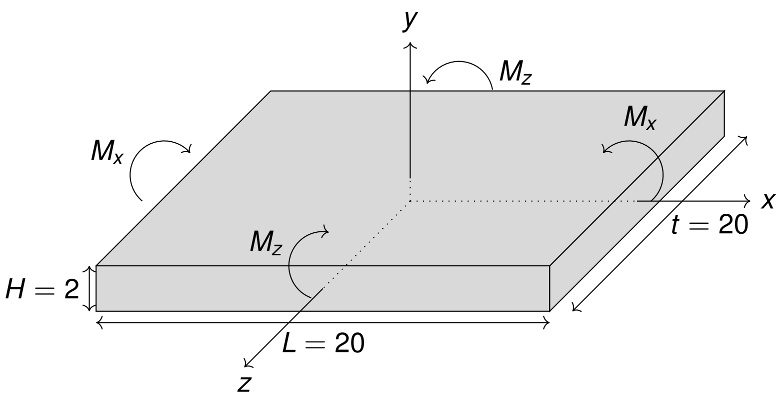
Length $L = 20$, height $H = 2$, and thickness $t = 20$. $E = 2500$, $\nu \in \{0.25, 0.4999\}$ (and $\mu = \frac{E}{2(1+\nu)}$, $\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$), $\mu_c = 0.5\mu$, $\alpha = 2\mu\, L_c^2$, $\beta = 2\mu\, L_c^2$, and $\gamma = 4\mu\, L_c^2$, with $L_c = 1$ the characteristic length. A bending moment $M_x = 100$ is applied on the left and right boundary. The exact solution is prescribed by

$$u_x = \frac{M_x xy}{D + \gamma H}, \qquad u_y = -\frac{M_x}{2(D + \gamma H)}\left(x^2 + \frac{\nu}{1-\nu} y^2\right) + \frac{M_x}{24(D + \gamma H)}\left(L^2 + \frac{\nu}{1-\nu} H^2\right),$$
$$\omega_z = -\frac{M_x x}{D + \gamma H},$$

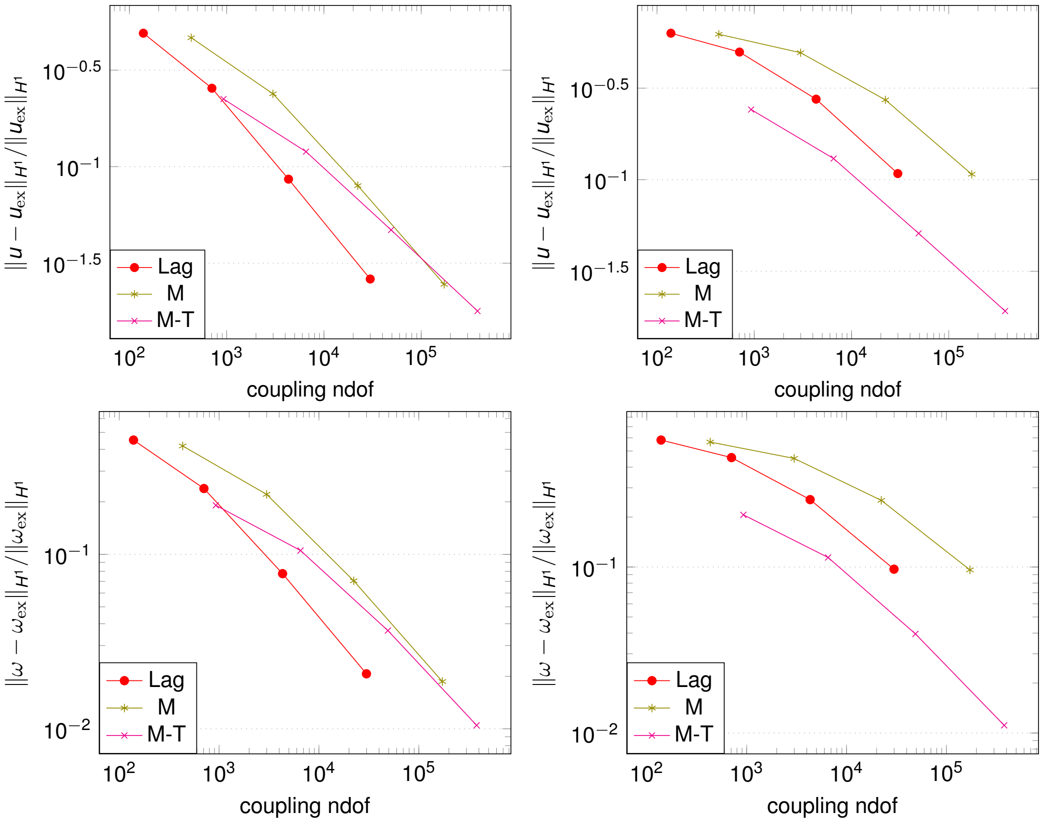
where $D = \frac{E\, H^3}{12(1-\nu^2)}$. The resulting non-zero stress components are

$$\sigma_{xx} = \frac{E}{1-\nu^2}\frac{M_x y}{D + \gamma H}, \qquad \sigma_{zz} = \frac{\nu\, E}{1-\nu^2}\frac{M_x y}{D + \gamma H}, \qquad m_{xz} = -\frac{\beta M_x}{D + \gamma H}, \qquad m_{zx} = -\frac{\gamma M_x}{D + \gamma H}$$

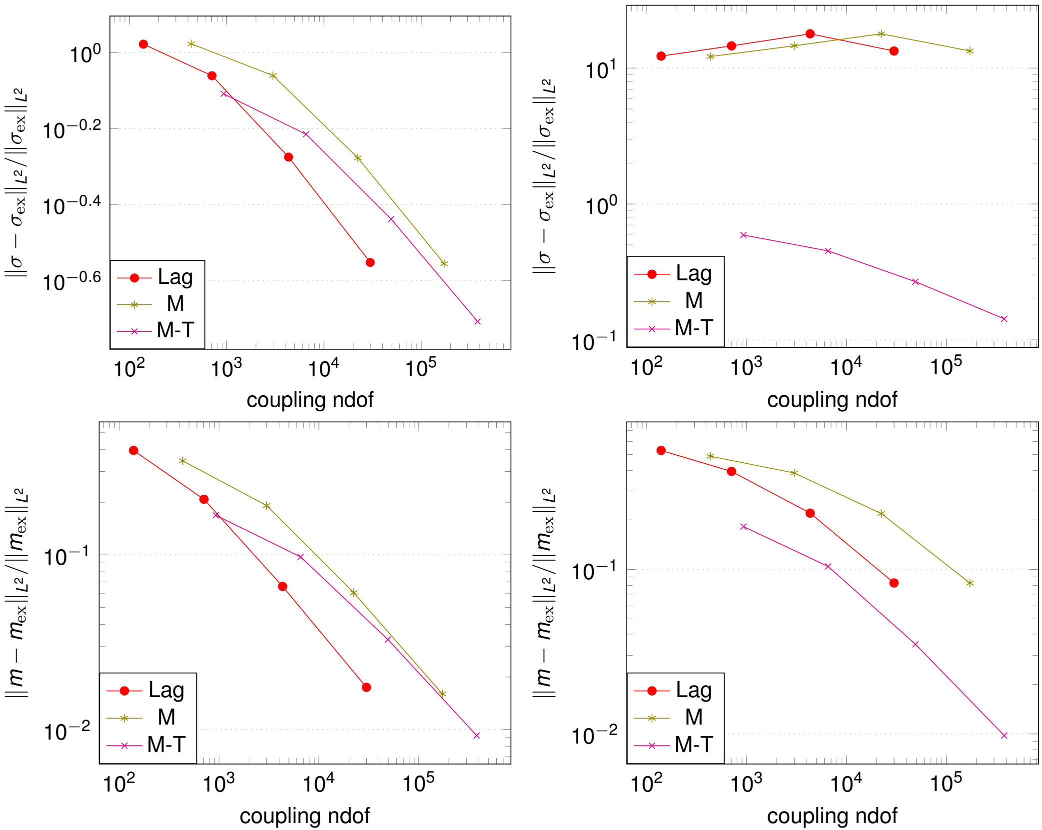
and no volume forces apply.



CONVERGENCE RATES. LEFT: $\nu = 0.25$. RIGHT: $\nu = 0.4999$.



Convergence rates. Left: $\nu = 0.25$. Right: $\nu = 0.4999$.



Numerical test: Torsion of a cylinder

$\mu = 15$, $\lambda = 1$, $\mu_c = 5$, and $\alpha = \beta = \gamma = 0.5$. Exact solution known.

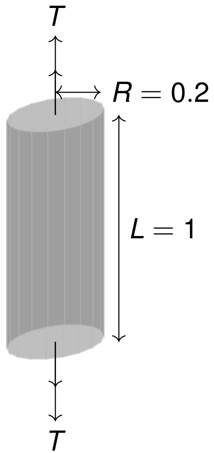


Figure. Geometry of torsion of cylinder example.

CONVERGENCE: DEGREE $k = 1$

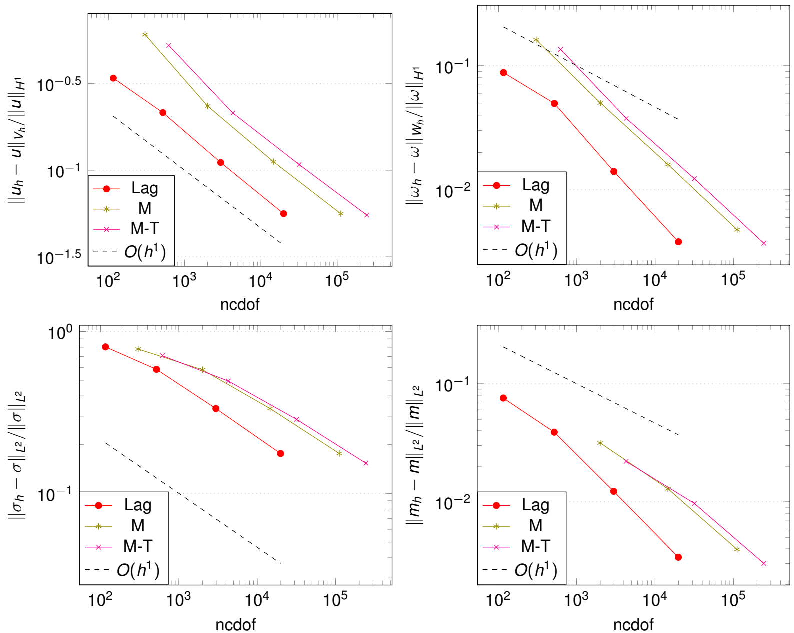


Figure. Convergence rates for cylinder torsion with $k = 1$. For M^1 and M-T^1 $\tilde{\omega}_h$ is used instead of ω_h.

Convergence: degree $k = 2$

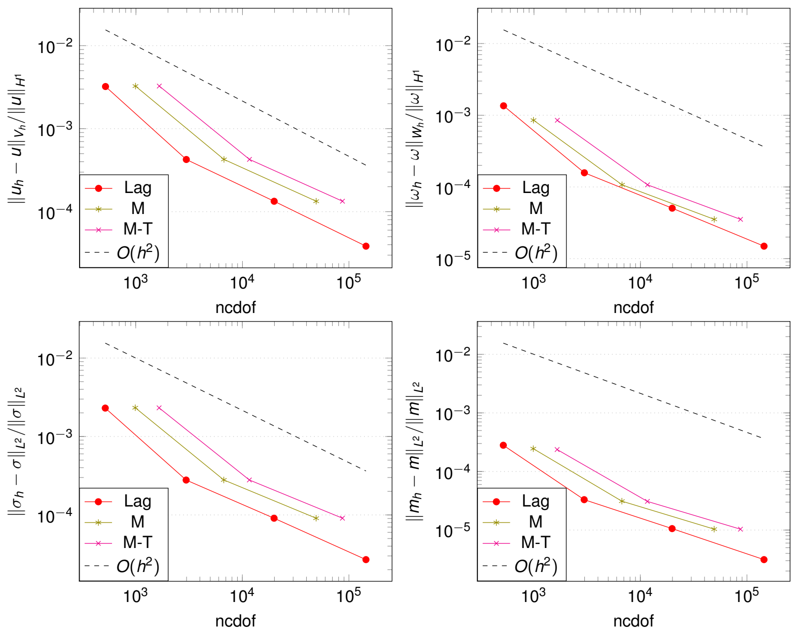


Figure. Convergence rates for cylinder torsion with $k = 2$. For M^2 and $M\text{-}T^2$ $\tilde{\omega}_h$ is used instead of ω_h.

ROBUSTNESS

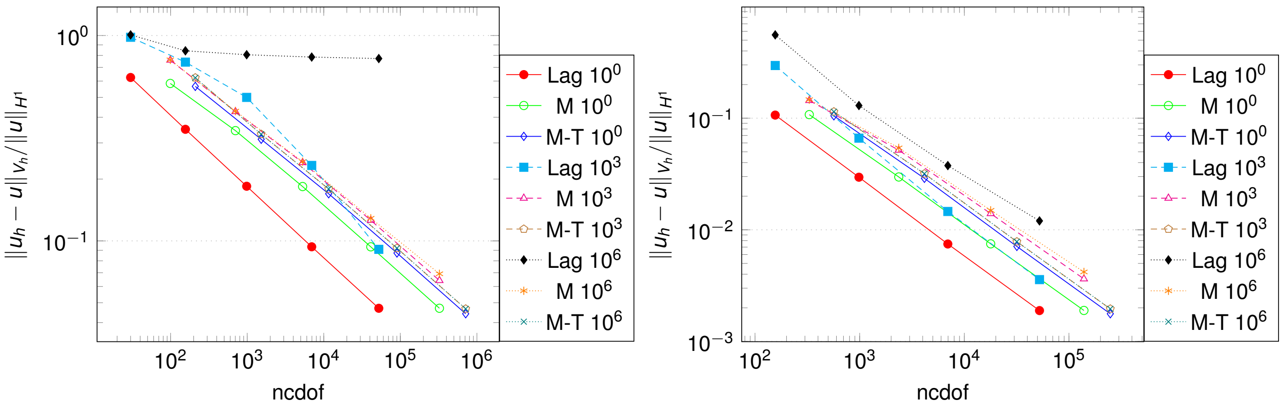


Figure. Results robustness test for $\mu_c/\mu \in \{1, 10^3, 10^6\}$ with methods of order $k = 1$ (left) and $k = 2$ (right).

Summary

- connections between continuum modelling, geometry and differential complexes (and thus analysis and numerics).
- discretizing models by discretizing entire complexes.

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