Many facets of cohomology

– Structure-aware formulations and structure-preserving discretisation –

Kaibo Hu

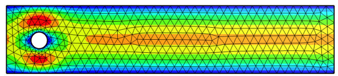
April 2025

Motivation: compatible discretisation

Fundamental question: how to discretise a system with more than one variable?

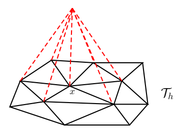
$$
\begin{aligned}
\int \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v} \, dx - \int p \nabla \cdot \boldsymbol{v} \, dx &= \int \boldsymbol{f} \cdot \boldsymbol{v} \, dx, \quad \forall \boldsymbol{v}, \\
\int \nabla \cdot \boldsymbol{u} q \, dx &= 0, \quad \forall q.
\end{aligned}
$$

Velocity continuous $\mathcal{P}_4$, pressure discontinuous $\mathcal{P}_3$

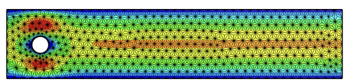


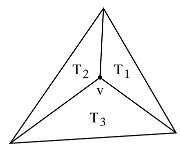
Velocity continuous $\mathcal{P}_2$, pressure discontinuous $\mathcal{P}_1$

```
---------------------------------------------------------------------------
NgException                               Traceback (most recent call last)
Cell In[9], line 6
      3 print ("V.ndof =", V.ndof, ", Q.ndof =", Q.ndof)
      4 X = V*Q
----> 6 gfu = SolveStokes(X)

Cell In[7], line 14, in SolveStokes(X)
     11 gfu.Set(uin, definedon=mesh.Boundaries("inlet"))
     13 res = -a.mat * gf.vec
---> 14 inv = a.mat.Inverse(freedofs=X.FreeDofs(), inverse="umfpack")
     15 gf.vec.data += inv * res
     17 Draw(gfu)

NgException: UmfpackInverse: Numeric factorization failed.
```



Velocity continuous $\mathcal{P}_2$, pressure discontinuous $\mathcal{P}_1$, on Alfeld split

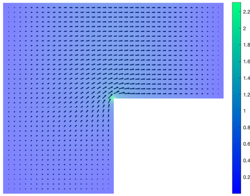




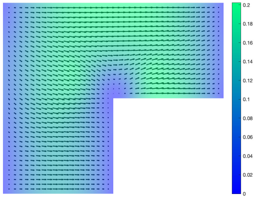
Numerical results obtained by NGSolve.

There may be no visible clues to tell spurious solutions.

$$-\operatorname{curl}\Delta\operatorname{rot}\boldsymbol{u}-\operatorname{grad}\operatorname{div}\boldsymbol{u}=\boldsymbol{f}.$$



scalar finite elements



“intrinsic” FEs

model problem for generalised continua, classical finite element converges to a wrong solution.

- K. Hu et al. *Spurious solutions for high order curl problems*, IMA Journal of Numerical Analysis (2023).

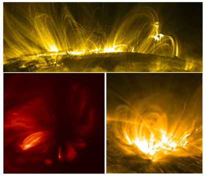
	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8
Intrinsic FEs	0.000000	0.593379	0.595179	1.801959	2.837796	4.458048	4.492200	5.463407
Scalar FEs	1.947637	2.579732	2.731537	3.781333	5.542562	7.373284	7.571471	7.797919

Table. Eigenvalues λ_1 to λ_8.

Many more examples available.

Motivation: structure-preserving discretisation

Fundamental question in plasma physics: given initial data, what does the system evolve to?
heating of solar corona, plasma equilibria (magnetic configurations) etc.



$$\begin{aligned} \boldsymbol{B}_t - \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) &= 0, \\ \boldsymbol{j} &= \nabla \times \boldsymbol{B}, \\ \boldsymbol{u} &= \tau \boldsymbol{j} \times \boldsymbol{B}. \end{aligned}$$

Energy decay

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{B}\|^2 = -\tau \|\boldsymbol{B} \times \boldsymbol{j}\|^2.$$

Helicity conservation

$$\frac{d}{dt}\mathcal{H}_m = 0, \quad \text{with } \mathcal{H}_m := \int \boldsymbol{A}\cdot\boldsymbol{B}\, dx, \quad \boldsymbol{B} = \nabla \times \boldsymbol{A}.$$

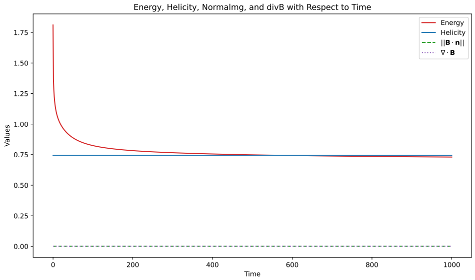


Figure. Helicity-preserving scheme

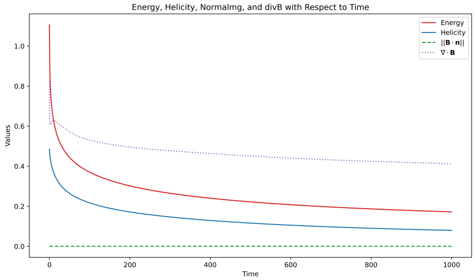


Figure. CG scheme (non-preserving)

- *Topology-preserving discretization for the magneto-frictional equations arising in the Parker conjecture*, M. He, P. E. Farrell, KH, B. Andrews, arXiv (2025).

Computation is used for computing gravitational wave templates, investigating magnetic configurations for fusion devices, designing quantum computing devices etc.

How confident are we in what we computation?

Key: *many facets of differential complexes and cohomology*, appearing in many problems in different forms.

Computation is used for computing gravitational wave templates , investigating magnetic configurations for fusion devices , designing quantum computing devices etc.

How confident are we in what we computation?

Key: *many facets of differential complexes and cohomology*, appearing in many problems in different forms.

$$\cdots \longrightarrow V^{k-1} \xrightarrow{d^{k-1}} V^k \xrightarrow{d^k} V^{k+1} \longrightarrow \cdots$$

$$0 \longrightarrow C^\infty(\Omega) \xrightarrow{\text{grad}} C^\infty(\Omega;\mathbb{R}^3) \xrightarrow{\text{curl}} C^\infty(\Omega;\mathbb{R}^3) \xrightarrow{\text{div}} C^\infty(\Omega) \longrightarrow 0.$$

$$d^0 := \text{grad}, \quad d^1 := \text{curl}, \quad d^2 := \text{div}\,.$$

- complex property: $d^k \circ d^{k-1} = 0, \ \Rightarrow \ \mathcal{R}(d^{k-1}) \subset \ker(d^k)$,
  $\text{curl} \circ \text{grad} = 0 \Rightarrow \mathcal{R}(\text{grad}) \subset \ker(\text{curl}), \quad \text{div} \circ \text{curl} = 0 \Rightarrow \mathcal{R}(\text{curl}) \subset \ker(\text{div})$
- cohomology: $\mathscr{H}^k := \ker(d^k)/\mathcal{R}(d^{k-1})$,
  $\mathscr{H}^0 := \ker(\text{grad}), \quad \mathscr{H}^1 := \ker(\text{curl})/\mathcal{R}(\text{grad}), \quad \mathscr{H}^2 := \ker(\text{div})/\mathcal{R}(\text{curl})$
- exactness: $\ker(d^k) = \mathcal{R}(d^{k-1})$, i.e., $d^k u = 0 \ \Rightarrow u = d^{k-1} v$
  $\text{curl}\, u = 0 \Rightarrow u = \text{grad}\,\phi, \quad \text{div}\, v = 0 \Rightarrow v = \text{curl}\,\psi.$

Solving equations is homological algebra: philosophical contemplation

Solving equations: given $g \in G$ and $\mathscr{L} : W \to G$, find $w \in W$, such that

$$\mathscr{L}(w) = g.$$

- Existence: surjecitvity of $\mathscr{L} : W \to G \iff$ exactness of

$$W \xrightarrow{\mathscr{L}} G \longrightarrow 0$$

- Stability: for $\forall g \in G$, $\exists w \in W$, such that $\mathscr{L} w = g$ and $\|w\|_W \leq C\|g\|_G$.

$$\inf_{g \in G} \sup_{w \in W} \frac{(\mathscr{L} w, g)}{\|w\|_W \|g\|_G} \geq \alpha > 0$$

- Uniqueness: injectivity of $\mathscr{L} : W \to G \iff$ exactness of

$$0 \longrightarrow W \xrightarrow{\mathscr{L}} G$$

SOLVING EQUATIONS IS HOMOLOGICAL ALGEBRA: PHILOSOPHICAL CONTEMPLATION

Solving equations: given $g \in G$ and $\mathscr{L} : W \to G$, find $w \in W$, such that

$$\mathscr{L}(w) = g.$$

- existence , uniqueness , stability : exactness + norm control

$$0 \longrightarrow W \xrightarrow{\mathscr{L}} G \longrightarrow 0$$

well-posed algorithms $\Longleftrightarrow$ schemes preserving cohomology

Other concepts, such as compatibility conditions and rigidity can be obtained in similar ways.

Solving equations: given $g \in G$ and $\mathscr{L} : W \to G$, find $w \in W$, such that

$$\mathscr{L}(w) = g.$$

- Compatibility conditions when existence does not hold:
  exactness of
  $$0 \longrightarrow W \xrightarrow{\mathscr{L}} G \xrightarrow{\mathscr{S}} Q \longrightarrow 0.$$
  For any g satisfying $\mathscr{S} g = 0$, $\exists w \in W$, such that $\mathscr{L} w = g$. Rigidity when uniqueness does not hold:

  exactness of
  $$0 \longrightarrow V \xrightarrow{\mathscr{T}} W \xrightarrow{\mathscr{L}} G \longrightarrow 0.$$
  $\mathscr{L} w = g$, w is unique up to elements in V.

Outline

Stokes Pairs and Compatible Discretisation

The Stokes problem:

$$\begin{aligned} \int \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v}\, dx - \int p \nabla \cdot \boldsymbol{v}\, dx &= \int \boldsymbol{f} \cdot \boldsymbol{v}\, dx, \quad \forall \boldsymbol{v}, \\ \int \nabla \cdot \boldsymbol{u}\, q\, dx &= 0, \quad \forall q. \end{aligned}$$

Continuous Level (PDEs):
Well-posedness via inf-sup condition:

$$\inf_{q \in L^2/\mathbb{R}} \sup_{\boldsymbol{v} \in \boldsymbol{H}_0^1} \frac{\int \operatorname{div} \boldsymbol{v} q\, dx}{\|\boldsymbol{v}\|_{H^1} \|q\|_{L^2}} \geq \gamma > 0$$

From exact de Rham complex:

$$0 \longrightarrow V^0 \xrightarrow{\text{grad}} V^1 \xrightarrow{\text{curl}} V^2 \xrightarrow{\text{div}} V^3 \longrightarrow 0$$

velocity pressure

Discrete Level (Numerics):
Stability via discrete inf-sup:

$$\inf_{q_h \in Q_h} \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{\int \operatorname{div} \boldsymbol{v}_h q_h\, dx}{\|\boldsymbol{v}_h\|_{H^1} \|q_h\|_{L^2}} \geq \gamma > 0$$

Achieved by discrete de Rham complex:

$$0 \longrightarrow \cdots \xrightarrow{\text{grad}} \cdots \xrightarrow{\text{curl}} \boldsymbol{V}_h \xrightarrow{\text{div}} Q_h \longrightarrow 0$$

velocity pressure

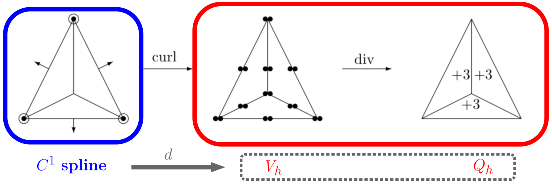
Constructing Finite Element Stokes Pair

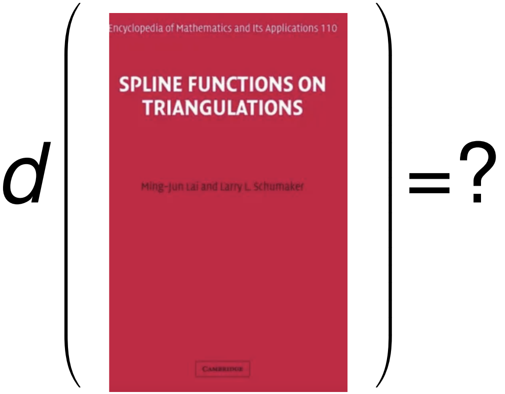
A Long-Standing Challenge

Construct velocity space $\boldsymbol{V}_h \subset \boldsymbol{H}^1$ and pressure space $Q_h \subset L^2$ such that div $\boldsymbol{V}_h = Q_h$.

Alfeld Split: Arnold-Qin 1992, Christiansen-KH 2018

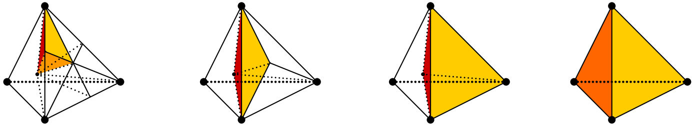
- Continuous $\mathcal{P}_2$, discontinuous $\mathcal{P}_1$
- $\mathcal{C}^1$ scalar spline on this triangulation
- Differentiating it yields the $\mathcal{P}_2$-$\mathcal{P}_1$ pair
- Ensures div $\boldsymbol{V}_h = Q_h$



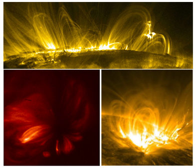


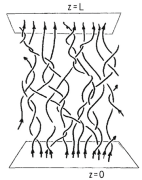
Christiansen-KH 2018: Systematic construction of Stokes complexes via scalar spline differentiation.

Christiansen, S. H., & Hu, K. (2018). Generalized finite element systems for smooth differential forms and Stokes' problem. *Numerische Mathematik*, 140, 327–371.



Ideal magnetic relaxation





Eugene Parker

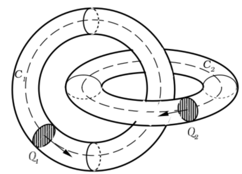
Parker hypothesis (Still Open)

For “almost any initial data”, *the magnetic field develops tangential discontinuities (current sheet) during the relaxation to static equilibrium.*

Finer structure: helicity [MHD: Woltjer’s invariant, ideal fluid: Moffatt (giving the name)]

$$\mathcal{H}_m := \int \boldsymbol{A} \cdot \boldsymbol{B} \, dx.$$

Describe *knots of divergence-free fields*. Conserved in ideal MHD.



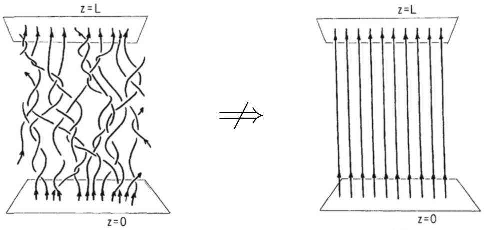
A TOPOLOGICAL MECHANISM

Arnold inequality (V.I. Arnold 1974): helicity provides lower bound for energy

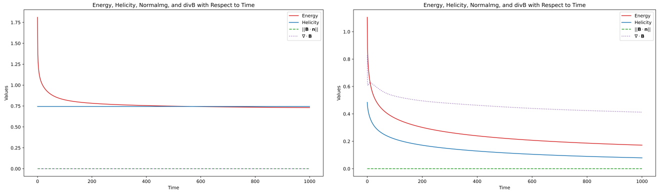
$$\left| \int \boldsymbol{A} \cdot \boldsymbol{B} \, dx \right| \leq C \int |\boldsymbol{B}|^2 \, dx$$

Proof. Cauchy-Schwarz $|\int \boldsymbol{A} \cdot \boldsymbol{B} \, dx| \leq \|\boldsymbol{A}\|_{L^2} \|\boldsymbol{B}\|_{L^2}$ + Poincaré inequality $\|\boldsymbol{A}\|_{L^2} \leq C \|\nabla \times \boldsymbol{A}\|_{L^2}$.

Vladimir Igorevich Arnold



Knots provide topological barriers preventing energy decay. This **mechanism** is lost in computation if algorithms do not preserve helicity.



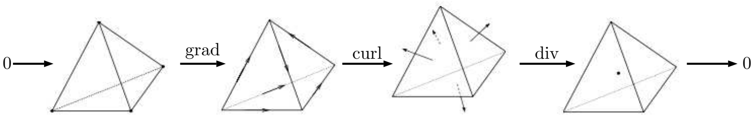
Patrick Farrell Mingdong He Boris Andrews

Topology-preserving discretization for the magneto-frictional equations arising in the Parker conjecture, M. He, P. E. Farrell, KH, B. Andrews, arXiv (2025).

How to preserve helicity: discrete de Rham complex

- Raviart–Thomas (1977), Nédélec (1980): Early finite elements
- Bossavit (1988): Differential forms and complex
- Hiptmair (1999), Arnold, Falk, Winther (2006): Systematic Finite Element Exterior Calculus

Classical Whitney forms



$\boldsymbol{E}_h, \boldsymbol{H}_h$ $\boldsymbol{B}_h$

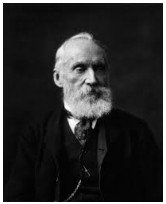
- Faraday's law $\partial_t \boldsymbol{B}_h + \nabla \times \boldsymbol{E}_h = 0$ holds precisely $\Longrightarrow \frac{d}{dt}(\nabla \cdot \boldsymbol{B}_h) = 0$.
- Introducing projection $\boldsymbol{H}_h = Q_{L^2}\boldsymbol{B}_h \quad \Longrightarrow \quad (\boldsymbol{u}_h \times \boldsymbol{H}_h, Q_{L^2}\boldsymbol{B}_h) = 0.$

First finite element method for MHD preserving $\nabla \cdot \boldsymbol{B} = 0$, energy & helicity:

KH,Hu,Ma,Xu 2016, KH,Ma,Xu 2017, KH,Lee,Xu 2021, Laakmann,Hu,Farrell 2023.

TOWARDS *Computational Topological Hydrodynamics*

A subject back to Kelvin, Helmholtz, and more recently by Arnold, Moffatt, Sullivan...
limited applications due to lack of topology-preserving algorithms

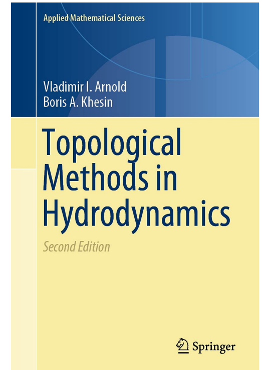

The Lord Kelvin


Hermann von Helmholtz


Vladimir Arnold


Keith Moffatt


Dennis Sullivan



> *Direct computational assessment of **Parker's hypothesis** brings a number of challenges. Foremost among these is the requirement to **precisely maintain the magnetic topology during the simulated evolution**, i.e., precisely maintain the magnetic field line mapping between the two line-tied boundaries. . . . In the following sections, two methods are described which seek to mitigate against these difficulties. However, in all cases the **representation of current singularities remains problematic**. . .*

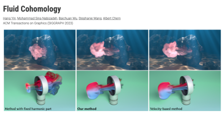
— *The Parker problem: existence of smooth force-free fields and coronal heating*, Pontin, Hornig, Living Rev. Sol. Phys. 2020.

Many Open Problems and Opportunities

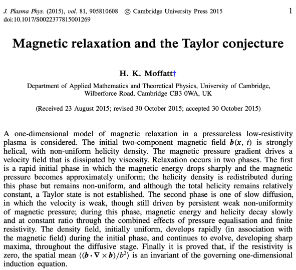
Towards *Computational Topological Hydrodynamics*

Fluid Cohomology

Theorem 3.5 ([Arn10]) *The number of linearly independent stationary k-forms is not less than the k^{th} Betti number b_k of the manifold M.*



Taylor's Conjecture



J. Plasma Phys. (2015), *vol.* 81, 905810608 © Cambridge University Press 2015 1
doi:10.1017/S0022377815001269

Magnetic relaxation and the Taylor conjecture

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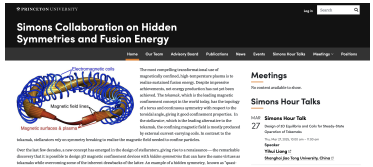
(Received 23 August 2015; revised 30 October 2015; accepted 30 October 2015)

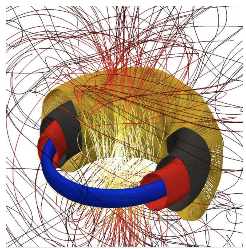
A one-dimensional model of magnetic relaxation in a pressureless low-resistivity plasma is considered. The initial two-component magnetic field $\boldsymbol{b}(x, t)$ is strongly helical, with non-uniform helicity density. The magnetic pressure gradient drives a velocity field that is dissipated by viscosity. Relaxation occurs in two phases. The first is a rapid initial phase in which the magnetic energy drops sharply and the magnetic pressure becomes approximately uniform; the helicity density is redistributed during this phase but remains non-uniform, and although the total helicity remains relatively constant, a Taylor state is not established. The second phase is one of slow diffusion, in which the velocity is weak, though still driven by persistent weak non-uniformity of magnetic pressure; during this phase, magnetic energy and helicity decay slowly and at constant ratio through the combined effects of pressure equalisation and finite resistivity. The density field, initially uniform, develops rapidly (in association with the magnetic field) during the initial phase, and continues to evolve, developing sharp maxima, throughout the diffusive stage. Finally it is proved that, if the resistivity is zero, the spatial mean $\langle(\boldsymbol{b}\cdot\nabla\times\boldsymbol{b})/b^2\rangle$ is an invariant of the governing one-dimensional induction equation.

Symmetry Reduction



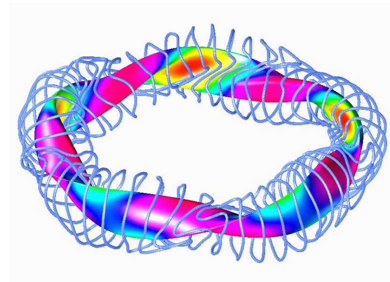
MHD, hidden symmetries, Lie group

Self-organisation



Plasma dynamics

Stellarator Optimization



3D field design

And More...

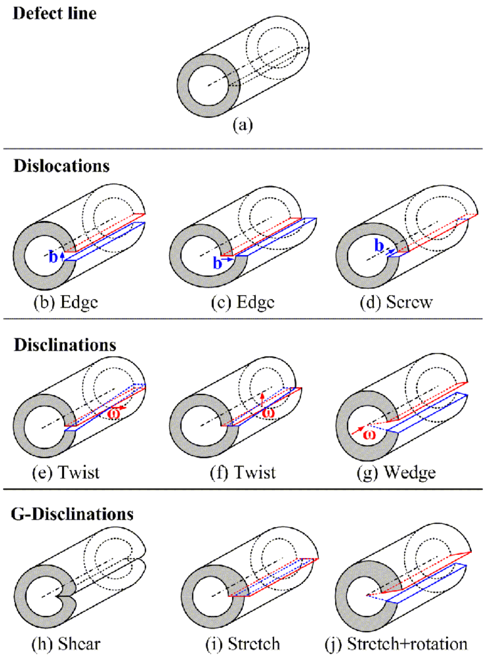
. . .

Emerging topics

Figures: by Chris Smiet; From "Scientific Visualization of 3-dimensional Optimized Stellarator Configurations," Donald A. Spong, Oak Ridge National Laboratory.

Solid mechanics: an Erlangen programme

A JUNGLE OF MODELS



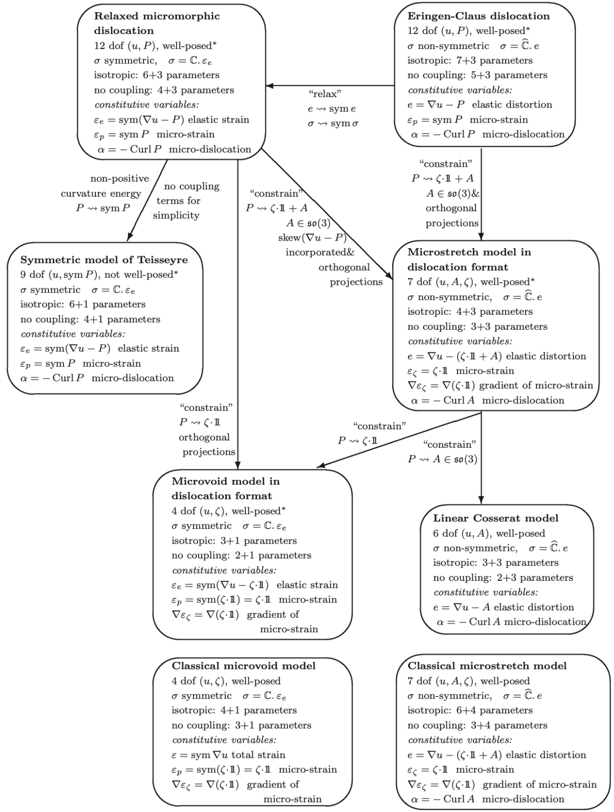


Figure 3: Relaxed micromorphic, microstretch model, Cosserat model, microstrain model and microvoid models in dislocation format

Voterra's seven distortion (defects) models and generalisations linear micromorphic continuum

Sun, X. Y. et al. (2017). Continuous description of grain boundaries using crystal defect fields: the example of a 3 1 0/[0 0 1] tilt boundary in MgO. European Journal of Mineralogy.

Neff, P., Ghiba, I. D., Madeo, A., Placidi, L., & Rosi, G. (2014). *A unifying perspective: the relaxed linear micromorphic continuum. Continuum Mechanics and Thermodynamics.*

Question: clarify *structures* behind the models to guide reliable and parameter-robust computation?

From here, we enter a world of tensors.

stress, strain tensors, dislocation density, disclination density in continuum mechanics,
metric, curvature (scalar, Ricci, Weyl, Riemann, Cotton...), torsion in differential geometry etc.

Question: clarify *structures* behind the models to guide reliable and parameter-robust computation?

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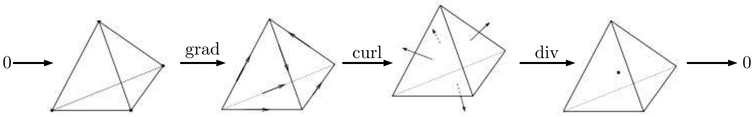
A special case: differential forms (fully skew-symmetric tensors), exterior derivatives



Whitney forms : the 2nd generation finite elements for forms (vectors) , following the 1st generation for scalars

Standard practice in computational electromagnetism and software.
e.g., Amazon's software for quantum computing hardware design

Question: What is *the* canonical discretisation for tensors – the 3rd generation finite elements ?

Differential structures in elasticity

Elasticity (Calabi, Kröner) complex

$$\mathrm{RM} \xrightarrow{\subset} C^\infty \otimes \mathbb{R}^3 \xrightarrow{\mathsf{sym\,grad}} C^\infty \otimes \mathbb{R}^{3\times 3}_{\mathsf{sym}} \xrightarrow{\mathsf{inc}} C^\infty \otimes \mathbb{R}^{3\times 3}_{\mathsf{sym}} \xrightarrow{\mathsf{div}} C^\infty \otimes \mathbb{R}^3 \longrightarrow 0$$

Differential structures in elasticity

Elasticity (Calabi, Kröner) complex

embedding $\mathbb{R}^3 \to \mathbb{R}^3$

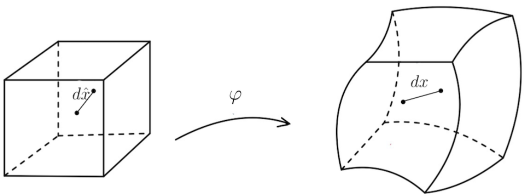
change of metric (strain)

$$\mathrm{RM} \xrightarrow{\subset} C^\infty \otimes \mathbb{R}^3 \xrightarrow{\text{sym grad}} C^\infty \otimes \mathbb{R}^{3\times 3}_{\text{sym}} \xrightarrow{\text{inc}} C^\infty \otimes \mathbb{R}^{3\times 3}_{\text{sym}} \xrightarrow{\text{div}} C^\infty \otimes \mathbb{R}^3 \longrightarrow 0$$

$$\varphi \longrightarrow \boldsymbol{e} = (\widehat{\nabla}\varphi)\cdot(\varphi\,\widehat{\nabla}) - I$$

$\boldsymbol{e} = 0$ iff φ is a rigid body motion.

Linearisation: $\boldsymbol{e} = \operatorname{sym}\operatorname{grad}\boldsymbol{u}$, in terms of displacement $\boldsymbol{u}(\widehat{\boldsymbol{x}}) = \varphi(\widehat{\boldsymbol{x}}) - \widehat{\boldsymbol{x}}$.



$$dx - d\hat{x} = d\hat{x}\cdot e\cdot d\hat{x}$$

DIFFERENTIAL STRUCTURES IN ELASTICITY

Elasticity (Calabi, Kröner) complex

metric (strain) | Riemann curvature

$$\mathrm{RM} \xrightarrow{\subset} C^\infty \otimes \mathbb{R}^3 \xrightarrow{\text{sym grad}} C^\infty \otimes \mathbb{R}^{3\times 3}_{\text{sym}} \xrightarrow{\text{inc}} C^\infty \otimes \mathbb{R}^{3\times 3}_{\text{sym}} \xrightarrow{\text{div}} C^\infty \otimes \mathbb{R}^3 \longrightarrow 0$$

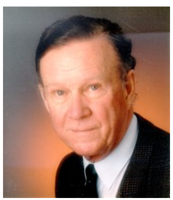
$$\boldsymbol{e} \longrightarrow \mathrm{Riem}(\boldsymbol{e})$$

Strain tensor (change of metric) $\boldsymbol{e} = (\widehat{\nabla}\boldsymbol{\varphi}) \cdot (\boldsymbol{\varphi}\,\widehat{\nabla}) - \boldsymbol{I}$ satisfies $\mathrm{Riem}(\boldsymbol{e}) = 0$.

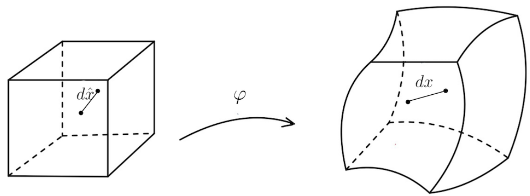
Defect theory: Kröner et al. used violation of compatibility conditions to model defects and incompatibility

Linearisation: Saint-Venant compatibility condition $\mathrm{inc}\,\boldsymbol{e} := \nabla \times \boldsymbol{e} \times \nabla = 0$.

Bernhard Riemann



Ekkehart Kröner



$$dx - d\hat{x} = d\hat{x} \cdot e \cdot d\hat{x}$$

Differential structures in elasticity

Elasticity (Calabi, Kröner) complex

curvature / stress

covector / force

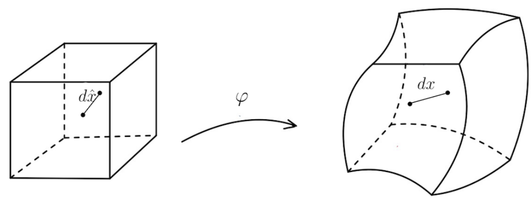
$$\mathrm{RM} \xrightarrow{\subset} C^\infty \otimes \mathbb{R}^3 \xrightarrow{\text{sym grad}} C^\infty \otimes \mathbb{R}^{3\times 3}_{\text{sym}} \xrightarrow{\text{inc}} C^\infty \otimes \mathbb{R}^{3\times 3}_{\text{sym}} \xrightarrow{\text{div}} C^\infty \otimes \mathbb{R}^3 \longrightarrow 0$$

$$\boldsymbol{\sigma} \longrightarrow \nabla \cdot \boldsymbol{\sigma}$$

Cauchy stress tensor $\boldsymbol{\sigma}$ balances load $\operatorname{div} \boldsymbol{\sigma} = \boldsymbol{f}$ with $\boldsymbol{\sigma} = A\boldsymbol{e}$ (Hooke's law); incompatibility causes internal stress $\operatorname{inc} \boldsymbol{e}$.

Robert Hooke
(Christ Church PDRA room)

Augustin-Louis Cauchy



$$dx - d\hat{x} = d\hat{x} \cdot e \cdot d\hat{x}$$

A NONLINEAR COMPLEX

cohomology not well defined, but exactness is. Exactness corresponds to important theorems.

Observations: KH, *Nonlinear elasticity complex and a finite element diagram chase* Springer INdAM Series (2024).

exactness: rigidity

two motions induce same metric iff up to RM

$$\text{rigid body motion} \xrightarrow{\subset} \text{map } \mathbb{R}^3 \text{ to } \mathbb{R}^3 \xrightarrow{\varphi \mapsto \varphi^* g_0 - g_0} \text{metric} \xrightarrow{\text{Ricci}} \text{curvature}$$

exactness: fundamental thm of Riem geometry

metric has vanishing curvature iff metric is Euclidean

Challenges for discretising nonlinear complex even in 1D:

$$0 \longrightarrow C^\infty \xrightarrow{u \mapsto u^2} C^\infty_+ \longrightarrow 0 \quad \text{exact: } \forall w, \exists u = \sqrt{w}, \text{ s.t., } w = u^2.$$

not work for polynomials!

$$0 \longrightarrow \mathcal{P}_k \xrightarrow{u \mapsto u^2} \mathcal{P}^+_{k-1} \longrightarrow 0$$

Algebraic geometric issues. Relevant to preserving nonlinear constraints.

Question: tools for studying nonlinear complexes? discretisation?

Complexes from complexes

Generating, analysing and discretising linear (deformation) complexes: *complexes from complexes*

- Douglas Arnold, KH, *Complexes from complexes*, Foundations of Computational Mathematics (2021) [1]

Step 1: connect two (or more) de Rham complexes

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R}^3 \arrow[r, "\mathrm{grad}"] & \mathbb{R}^{3\times 3} \arrow[r, "\mathrm{curl}"] & \mathbb{R}^{3\times 3} \arrow[r, "\mathrm{div}"] & \mathbb{R}^3 \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{R}^3 \arrow[r, "\mathrm{grad}"] \arrow[ur, "S^0"] & \mathbb{R}^{3\times 3} \arrow[r, "\mathrm{curl}"] \arrow[ur, "S^1"] & \mathbb{R}^{3\times 3} \arrow[r, "\mathrm{div}"] \arrow[ur, "S^2"] & \mathbb{R}^3 \arrow[r] & 0
\end{tikzcd}$$

$S^\bullet$: algebraic operators, connecting components of vectors/matrices

[1] Frontiers of Science Award 2025

Complexes from complexes

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Step 2: elimination

$$
\begin{tikzcd}
0 \arrow[r] & \mathbb{R}^3 \arrow[r, "\mathrm{grad}"] & \mathbb{S}+\mathbb{K} \arrow[r, "\mathrm{curl}"] & \mathbb{R}^{3\times 3} \arrow[r, "\mathrm{div}"] & \mathbb{R}^3 \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{R}^3 \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\mathrm{mskw}"] & \mathbb{R}^{3\times 3} \arrow[r, "\mathrm{curl}"] \arrow[ur, "\mathcal{S}"] & \mathbb{S}+\mathbb{K} \arrow[r, "\mathrm{div}"] \arrow[ur, "2\,\mathrm{vskw}"] & \mathbb{R}^3 \arrow[r] & 0
\end{tikzcd}
$$

$\mathbb{S}$: symmetric matrix, $\mathbb{K}$: skew-symmetric matrix

[1] Frontiers of Science Award 2025

Complexes from complexes

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Step 2: elimination

$$
\begin{tikzcd}
0 \arrow[r] & \mathbb{R}^3 \arrow[r, "\mathrm{grad}"] & \mathbb{S}+\cancel{\mathbb{K}} \arrow[r, "\mathrm{curl}"] & \cancel{\mathbb{R}^{3\times 3}} \arrow[r, "\mathrm{div}"] & \cancel{\mathbb{R}^3} \arrow[r] & 0 \\
0 \arrow[r] & \cancel{\mathbb{R}^3} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\,\mathrm{mskw}"] & \cancel{\mathbb{R}^{3\times 3}} \arrow[r, "\mathrm{curl}"] \arrow[ur, "\mathcal{S}"] & \mathbb{S}+\cancel{\mathbb{K}} \arrow[r, "\mathrm{div}"] \arrow[ur, "2\,\mathrm{vskw}"] & \mathbb{R}^3 \arrow[r] & 0
\end{tikzcd}
$$

$\mathbb{S}$: symmetric matrix, $\mathbb{K}$: skew-symmetric matrix

[1] Frontiers of Science Award 2025

COMPLEXES FROM COMPLEXES

Generating, analysing and discretising linear (deformation) complexes: *complexes from complexes*

- Douglas Arnold, KH, *Complexes from complexes*, Foundations of Computational Mathematics (2021) [1]

Step 3: connect rows by zig-zag

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R}^3 \arrow[r, "\text{sym grad}"] & \mathbb{S} \arrow[r, "\text{curl}"] & {} \arrow[dll] & & & \\
& & {} \arrow[r, "\text{curl}^{\mathrm{T}}"] & \mathbb{S} \arrow[r, "\text{div}"] & \mathbb{R}^3 \arrow[r] & 0.
\end{tikzcd}$$

Conclusion: cohomology of the output (elasticity) is isomorphic to the input (de Rham)

Analytic results follow: Poincaré–Korn inequalities, Hodge decomposition, compactness...

Inspired by the Bernstein-Gelfand-Gelfand (BGG) construction (B-G-G 1975, Čap,Slovák,Souček 2001, Eastwood 2000, Arnold,Falk,Winther 2006, Arnold, KH 2021, Čap, KH 2023)

[1] Frontiers of Science Award 2025

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& & {} \arrow[r, "\text{curl}^T"] & \mathbb{S} \arrow[r, "\text{div}"] & \mathbb{R}^3 \arrow[r] & 0.
\end{tikzcd}
$$

Conclusion: cohomology of the output (elasticity) is isomorphic to the input (de Rham)

Analytic results follow: Poincaré–Korn inequalities, Hodge decomposition, compactness...

But, is it purely mathematical?

[1] Frontiers of Science Award 2025

Solid mechanics: an Erlangen programme

embedding/displacement

metric/strain

$$\mathbb{R}^3 \xrightarrow{\text{sym grad}} \mathbb{S} \xrightarrow{\text{curl}\; T\; \text{curl}} \mathbb{S} \xrightarrow{\text{div}} \mathbb{R}^3$$

curvature/stress

covector/load

Riemann, Kröner, Cauchy, Hooke

Solid mechanics: an Erlangen programme

embedding/displacement

metric/strain

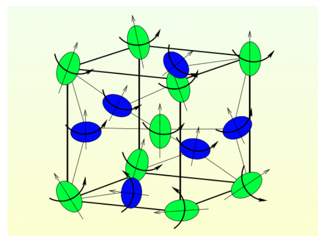
$$
\begin{tikzcd}[column sep=huge]
\mathbb{R}^3 \arrow[r, "\mathrm{grad}"] & \mathbb{R}^{3\times 3} \arrow[r, "\mathrm{curl}"] & {} \arrow[dl, "T"] & \\
\mathbb{R}^3 \arrow[ur, "S^0"] \arrow[r, "\mathrm{grad}"] & {} \arrow[r, "\mathrm{curl}"] & \mathbb{S} \arrow[r, "\mathrm{div}"] & \mathbb{R}^3
\end{tikzcd}
$$

rotation

curvature/stress

covector/load


Cosserat brothers



Cosserat continua: microstructures (rotation, stretch etc.)

Observations: A. Čap & KH, *BGG sequences with weak regularity and applications.* FoCM (2024).

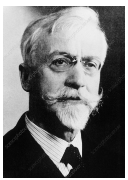
Leading to first parameter-robust scheme for Cosserat model: A.Dziubek, KH, M.Karow & M. Neunteufel, arXiv (2024).

Solid mechanics: an Erlangen programme

embedding/displacement | coframes | torsion 2-form dislocation | Bianchi

$$\begin{tikzcd}[column sep=huge]
\mathbb{R}^3 \arrow[r, "\mathrm{grad}"] & \mathbb{R}^{3\times 3} \arrow[r, "\mathrm{curl}"] & \mathbb{R}^{3\times 3} \arrow[r, "\mathrm{div}"] & \mathbb{R}^3 \\
\mathbb{R}^3 \arrow[r, "\mathrm{grad}"'] \arrow[ur, "S^0"] & \mathbb{R}^{3\times 3} \arrow[r, "\mathrm{curl}"'] \arrow[ur, "S^1"] & \mathbb{R}^{3\times 3} \arrow[r, "\mathrm{div}"'] \arrow[ur, "S^2"] & \mathbb{R}^3
\end{tikzcd}$$

rotation | connection 1-form | curvature/stress disclination | covector/load



Élie Cartan

Arash Yavari

Alain Goriely

Cartan's bridge between Einstein and Cosserat brothers – torsion

$$\begin{tikzcd}[column sep=tiny]
& \begin{array}{c}\text{Riemann-Cartan Manifolds } (\mathcal{B}, \nabla, \mathbf{G}) \\ \nabla\mathbf{G} = \mathbf{0}, \mathbf{T} \neq \mathbf{0}, \boldsymbol{\mathcal{R}} \neq \mathbf{0}\end{array} \arrow[dl] \arrow[dr] & \\
\begin{array}{c}\text{Riemannian Manifolds } (\mathcal{B}, \nabla, \mathbf{G}) \\ \nabla\mathbf{G} = \mathbf{0}, \mathbf{T} = \mathbf{0}, \boldsymbol{\mathcal{R}} \neq \mathbf{0}\end{array} \arrow[dr] & & \begin{array}{c}\text{Weitzenböck Manifolds } (\mathcal{B}, \nabla, \mathbf{G}) \\ \nabla\mathbf{G} = \mathbf{0}, \mathbf{T} \neq \mathbf{0}, \boldsymbol{\mathcal{R}} = \mathbf{0}\end{array} \arrow[dl] \\
& \begin{array}{c}\text{Flat(Euclidean)Manifolds } (\mathcal{B}, \nabla, \mathbf{G}) \\ \nabla\mathbf{G} = \mathbf{0}, \mathbf{T} = \mathbf{0}, \boldsymbol{\mathcal{R}} = \mathbf{0}\end{array} &
\end{tikzcd}$$

Riemann-Cartan Geometry of Nonlinear Dislocation Mechanics, Yavari and Goriely, ARMA (2012)

Observations: Christiansen, KH, & Lin, *Extended Regge complex for linearized Riemann-Cartan geometry and cohomology*. arXiv (2023). BGG construction is thus cohomology-preserving elimination of microstructures!

Solid mechanics: an Erlangen programme

In this way and more broadly, we develop structure-aware and computation-friendly modelling via complexes .
microstructures, defects, dimension reduction, contact mechanics, porous media...

Our 'BGG construction' is much broader. e.g. A generalisation to form-valued forms (double forms à ala Cartan)

$$
\begin{tikzcd}
 & \cdots \arrow[r] & \cdots \arrow[r] & \cdots \arrow[r] & \cdots & \\
\cdots \arrow[r] & \mathrm{Alt}^{k-1,\ell-1} \arrow[r, "d^{k-1}"] \arrow[ur] & \mathrm{Alt}^{k,\ell-1} \arrow[r, "d^{k}"] \arrow[ur] & \mathrm{Alt}^{k+1,\ell-1} \arrow[r, "d^{k+1}"] \arrow[ur] & \mathrm{Alt}^{k+2,\ell-1} \arrow[r] & \cdots \\
\cdots \arrow[r] & \mathrm{Alt}^{k-1,\ell} \arrow[r, "d^{k-1}"] \arrow[ur, "S^{k-1,\ell}"] & \mathrm{Alt}^{k,\ell} \arrow[r, "d^{k}"] \arrow[ur, "S^{k,\ell}"] & \mathrm{Alt}^{k+1,\ell} \arrow[r, "d^{k+1}"] \arrow[ur, "S^{k+1,\ell}"] & \mathrm{Alt}^{k+2,\ell} \arrow[r] & \cdots \\
\cdots \arrow[r] & \mathrm{Alt}^{k-1,\ell+1} \arrow[r, "d^{k-1}"] \arrow[ur, "S^{k-1,\ell+1}"] & \mathrm{Alt}^{k,\ell+1} \arrow[r, "d^{k}"] \arrow[ur, "S^{k,\ell+1}"] & \mathrm{Alt}^{k+1,\ell+1} \arrow[r, "d^{2}"] \arrow[ur, "S^{k+1,\ell+1}"] & \mathrm{Alt}^{k+2,\ell+1} \arrow[r] & \cdots \\
\cdots \arrow[r] & \mathrm{Alt}^{k-1,\ell+2} \arrow[r, "d^{k-1}"] \arrow[ur, "S^{k-1,\ell+2}"] & \mathrm{Alt}^{k,\ell+2} \arrow[r, "d^{k}"] \arrow[ur, "S^{k,\ell+2}"] & \mathrm{Alt}^{k+1,\ell+2} \arrow[r, "d^{k+1}"] \arrow[ur, "S^{k+1,\ell+2}"] & \mathrm{Alt}^{k+2,\ell+2} \arrow[r] & \cdots \\
 & \cdots \arrow[r] \arrow[ur] & \cdots \arrow[r] \arrow[ur] & \cdots \arrow[r] \arrow[ur] & \cdots &
\end{tikzcd}
$$

$\mathrm{Alt}^{k,\ell} := \mathrm{Alt}^k \otimes \mathrm{Alt}^\ell$: ℓ-form-valued k-forms

differential forms: $(k, 0)$ metric, strain: $(1, 1)$ curvature, stress: $(2, 2)$ torsion: $(2, 1)$

Solid mechanics: an Erlangen programme

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\cdots \arrow[r] & \mathrm{Alt}^{k-1,\ell} \arrow[r, "d^{k-1}"] \arrow[ur, "S^{k-1,\ell}"'] & \mathrm{Alt}^{k,\ell} \arrow[r, "d^{k}"] \arrow[ur, "S^{k,\ell}"'] & \mathrm{Alt}^{k+1,\ell} \arrow[r, "d^{k+1}"] \arrow[ur, "S^{k+1,\ell}"'] & \mathrm{Alt}^{k+2,\ell} \arrow[r] & \cdots \\
\cdots \arrow[r] & \mathrm{Alt}^{k-1,\ell+1} \arrow[r, "d^{k-1}"] \arrow[ur, "S^{k-1,\ell+1}"'] & \mathrm{Alt}^{k,\ell+1} \arrow[r, "d^{k}"] \arrow[ur, "S^{k,\ell+1}"'] & \mathrm{Alt}^{k+1,\ell+1} \arrow[r, "d^{2}"] \arrow[ur, "S^{k+1,\ell+1}"'] & \mathrm{Alt}^{k+2,\ell+1} \arrow[r] & \cdots \\
\cdots \arrow[r] & \mathrm{Alt}^{k-1,\ell+2} \arrow[r, "d^{k-1}"] \arrow[ur, "S^{k-1,\ell+2}"'] & \mathrm{Alt}^{k,\ell+2} \arrow[r, "d^{k}"] \arrow[ur, "S^{k,\ell+2}"'] & \mathrm{Alt}^{k+1,\ell+2} \arrow[r, "d^{k+1}"] \arrow[ur, "S^{k+1,\ell+2}"'] & \mathrm{Alt}^{k+2,\ell+2} \arrow[r] & \cdots \\
 & \cdots \arrow[r] \arrow[ur] & \cdots \arrow[r] \arrow[ur] & \cdots \arrow[r] \arrow[ur] & \cdots &
\end{tikzcd}
$$

$\mathrm{Alt}^{k,\ell} := \mathrm{Alt}^k \otimes \mathrm{Alt}^\ell$: ℓ-form-valued k-forms

differential forms: $(k, 0)$ metric, strain: $(1, 1)$ curvature, stress: $(2, 2)$ torsion: $(2, 1)$

Questions: Canonical discretisation of double forms?

Towards a finite element periodic table for tensors

KH, Ting Lin. *Finite element form-valued forms (I): Construction.* arXiv: 2503.03243 (2025)

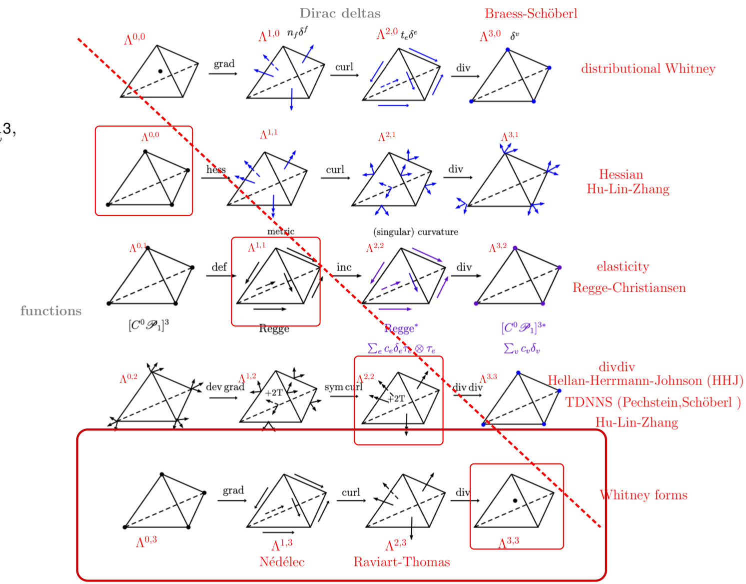
$$\mathrm{Alt}^{-1,-1} \dashrightarrow \quad \mathrm{Alt}^{0,0} \longrightarrow \mathrm{Alt}^{1,0} \longrightarrow \mathrm{Alt}^{2,0} \longrightarrow \mathrm{Alt}^{3,}$$

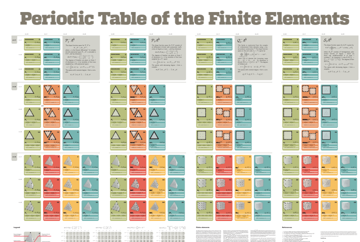
$$\mathrm{Alt}^{0,0} \longrightarrow \quad \longrightarrow \mathrm{Alt}^{1,1} \longrightarrow \mathrm{Alt}^{2,1} \longrightarrow \mathrm{Alt}^{3,1}$$

$$\mathrm{Alt}^{0,1} \longrightarrow \mathrm{Alt}^{1,1} \longrightarrow \quad \longrightarrow \mathrm{Alt}^{2,2} \longrightarrow \mathrm{Alt}^{3,2}$$

$$\mathrm{Alt}^{0,2} \longrightarrow \mathrm{Alt}^{1,2} \longrightarrow \mathrm{Alt}^{2,2} \longrightarrow \quad \longrightarrow \mathrm{Alt}^{3,3}$$

$$\mathrm{Alt}^{0,3} \longrightarrow \mathrm{Alt}^{1,3} \longrightarrow \mathrm{Alt}^{2,3} \longrightarrow \mathrm{Alt}^{3,3} \dashrightarrow \quad \mathrm{Alt}^{4,4}$$





classical Finite Element Exterior Calculus

Nédélec, Raviart–Thomas, Whitney, Bossavit, Hiptmair, Arnold, Falk, Winther...

TOWARDS A FINITE ELEMENT PERIODIC TABLE FOR TENSORS

KH, TING LIN. *Finite element form-valued forms (I): Construction.* ARXIV: 2503.03243 (2025)

$\mathrm{Alt}^{-1,-1}$

$\mathrm{Alt}^{0,0} \longrightarrow \mathrm{Alt}^{1,0} \longrightarrow \mathrm{Alt}^{2,0} \longrightarrow \mathrm{Alt}^{3,}$

$\mathrm{Alt}^{0,0} \longrightarrow$

$\longrightarrow \mathrm{Alt}^{1,1} \longrightarrow \mathrm{Alt}^{2,1} \longrightarrow \mathrm{Alt}^{3,1}$

$\mathrm{Alt}^{0,1} \longrightarrow \mathrm{Alt}^{1,1} \longrightarrow$

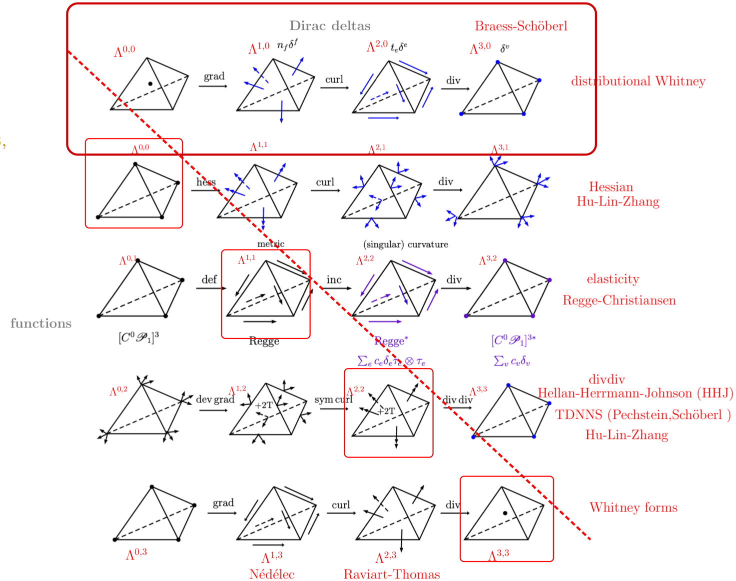
$\longrightarrow \mathrm{Alt}^{2,2} \longrightarrow \mathrm{Alt}^{3,2}$

$\mathrm{Alt}^{0,2} \longrightarrow \mathrm{Alt}^{1,2} \longrightarrow \mathrm{Alt}^{2,2} \longrightarrow$

$\longrightarrow \mathrm{Alt}^{3,3}$

$\mathrm{Alt}^{0,3} \longrightarrow \mathrm{Alt}^{1,3} \longrightarrow \mathrm{Alt}^{2,3} \longrightarrow \mathrm{Alt}^{3,3}$

$\mathrm{Alt}^{4,4}$



distributional de Rham complex (currents).

Braess, Schöberl 2008: equilibrated residual error estimator

Dietrich Braess

Joachim Schöberl

Towards a finite element periodic table for tensors

KH, Ting Lin. *Finite element form-valued forms (I): Construction.* arXiv: 2503.03243 (2025)

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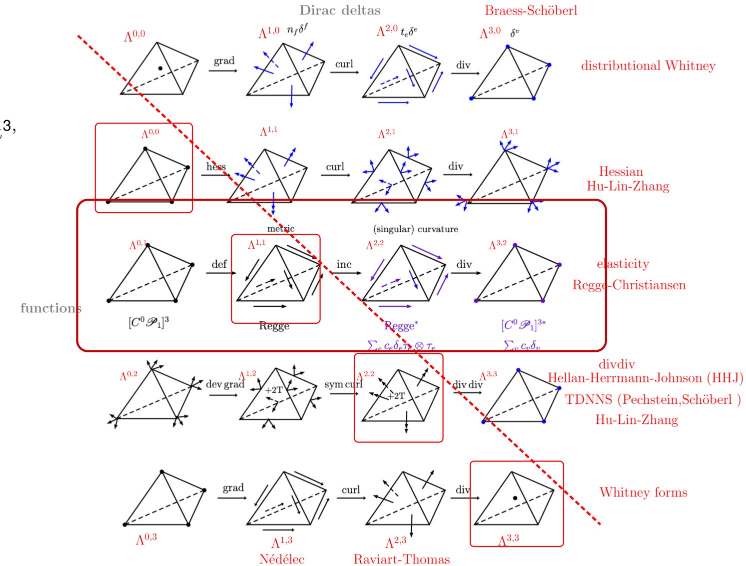

Christiansen's interpretation of Regge calculus as finite elements

Regge calculus (quantum & numerical gravity) :
edge length as metric, angle deficit as curvature

Regge finite element : piecewise constant symmetric tensor field


Tullio Regge


Snorre Christiansen

Towards a finite element periodic table for tensors

KH, Ting Lin. *Finite element form-valued forms (I): Construction.* arXiv: 2503.03243 (2025)

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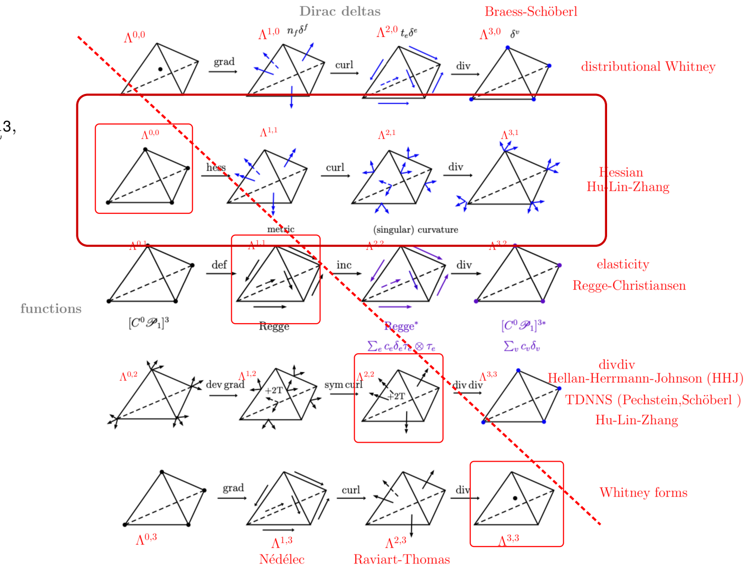
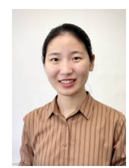
$$\mathrm{Alt}^{0,3} \longrightarrow \mathrm{Alt}^{1,3} \longrightarrow \mathrm{Alt}^{2,3} \longrightarrow \mathrm{Alt}^{3,3} \dashrightarrow \mathrm{Alt}^{4,4}$$



Hessian complex, unified structures identified.

Kaibo Hu, Ting Lin, Qian Zhang. *Distributional Hessian and divdiv complexes on triangulation and cohomology*. SIAM Journal on Applied Algebra and Geometry (2025).

Ting Lin



Qian Zhang

Towards a finite element periodic table for tensors

KH, Ting Lin. *Finite element form-valued forms (I): Construction.* arXiv: 2503.03243 (2025)

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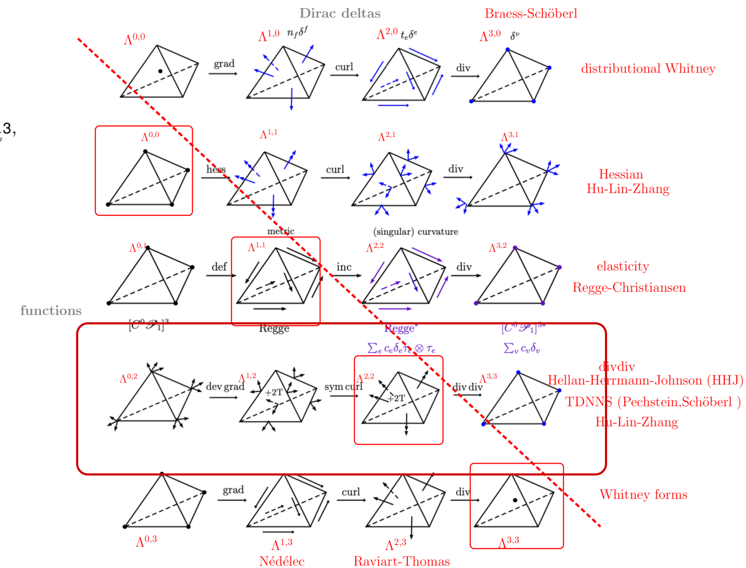


divdiv complex , dual to Hessian complex.

TDNNS for elasticity (Schöberl, Sinwel 2007), Hellan-Herrmann-Johnson (HHJ) element for plate.

Implemented by J.Schöberl in NGSolve with relativity applications

KH, Ting Lin, Qian Zhang. *Distributional Hessian and divdiv complexes on triangulation and cohomology*. SIAM Journal on Applied Algebra and Geometry (2025).

Astrid Pechstein

Joachim Schöberl

Towards a finite element periodic table for tensors

KH, Ting Lin. *Finite element form-valued forms (I): Construction.* arXiv: 2503.03243 (2025)

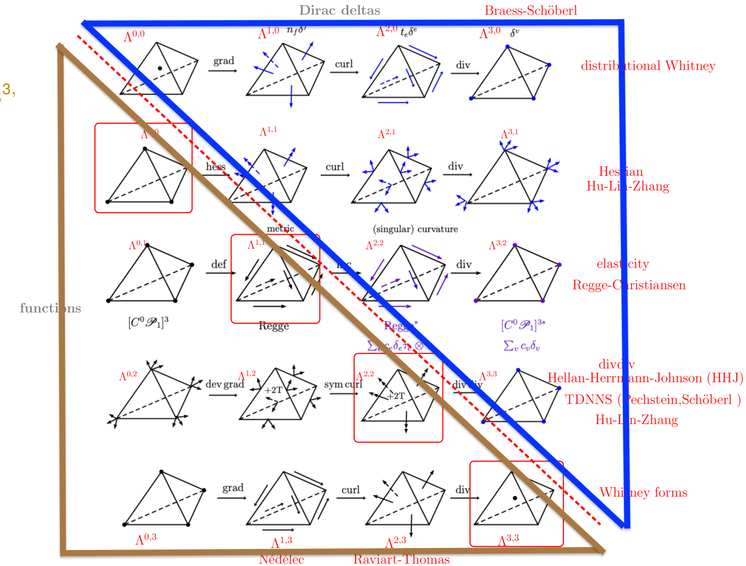
$$\mathrm{Alt}^{-1,-1} \dashrightarrow \quad \mathrm{Alt}^{0,0} \longrightarrow \mathrm{Alt}^{1,0} \longrightarrow \mathrm{Alt}^{2,0} \longrightarrow \mathrm{Alt}^{3,}$$

$$\mathrm{Alt}^{0,0} \longrightarrow \swarrow \longrightarrow \mathrm{Alt}^{1,1} \longrightarrow \mathrm{Alt}^{2,1} \longrightarrow \mathrm{Alt}^{3,1}$$

$$\mathrm{Alt}^{0,1} \longrightarrow \mathrm{Alt}^{1,1} \longrightarrow \swarrow \longrightarrow \mathrm{Alt}^{2,2} \longrightarrow \mathrm{Alt}^{3,2}$$

$$\mathrm{Alt}^{0,2} \longrightarrow \mathrm{Alt}^{1,2} \longrightarrow \mathrm{Alt}^{2,2} \longrightarrow \swarrow \longrightarrow \mathrm{Alt}^{3,3}$$

$$\mathrm{Alt}^{0,3} \longrightarrow \mathrm{Alt}^{1,3} \longrightarrow \mathrm{Alt}^{2,3} \longrightarrow \mathrm{Alt}^{3,3} \dashrightarrow \quad \mathrm{Alt}^{4,4}$$



Patterns, Symmetries, Duality.

functions (classical finite elements) v.s. Dirac measures (currents).   any dimension, any degree.

Georges de Rham

Classical Finite Element Periodic Table (last row) is the special case of the generalised Table where all spaces are finite elements in the classical sense.

General relativity: numerical analysis as a tool for discovery

spacetime geometry matter

$$G_{\alpha\beta} = \frac{8\pi}{c^4} T_{\alpha\beta}$$

Numerically solving the Einstein equations (numerical relativity) has been used to compute templates of gravitational waves and investigate new theories of gravity.

Connection from metric:

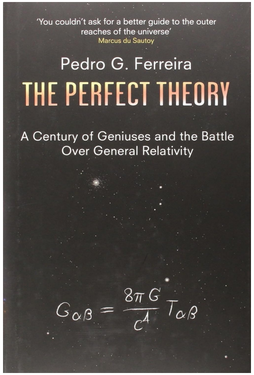
$$\Gamma^k_{ij} = g^{k\ell}\left(\frac{\partial g_{\ell i}}{\partial x^j} + \frac{\partial g_{\ell j}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^\ell}\right),$$

Riemannian tensor from connection:

$$R^\ell{}_{ijk} = \frac{\partial \Gamma^\ell_{ik}}{\partial x^j} - \frac{\partial \Gamma^\ell_{ij}}{\partial x^k} + \Gamma^\ell_{jm}\Gamma^m_{ik} - \Gamma^\ell_{km}\Gamma^m_{ij}.$$

Ricci tensor is the trace of Riemann: $R_{ik} = R^\ell{}_{i\ell k}$;
Einstein tensor is Ricci with modified trace:

$$G_{ik} = R_{ik} - \frac{1}{2} R g_{ik},$$



Pedro Fereirra

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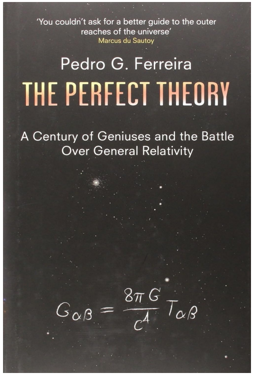
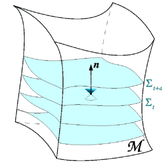


Pedro Fereirra

How do we trust our computation?

3+1 Einstein equations (ADM form):
[γ: 3-metric in 3+1 decomposition, α, β: lapse & shift (gauge freedom)]



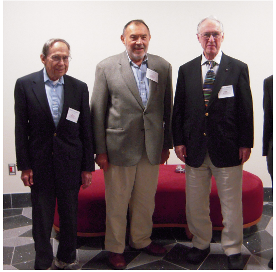
$$\partial_t \gamma_{ij} = -2\,\alpha\, K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i,$$ evolution of 3-metric

$$\partial_t K_{ij} = -\nabla_i \nabla_j\, \alpha + \alpha\big({}^{(3)}R_{ij} + K\, K_{ij} - 2\, K_{i\ell}\, K_j{}^{\ell}\big) + \beta^{\ell} \nabla_{\ell} K_{ij} + K_{\ell i} \nabla_j \beta^{\ell} + K_{\ell j} \nabla_i \beta^{\ell},$$ evolution of extrinsic curvature (embedding)

$$\mathcal{H} := {}^{(3)}R + K^2 - K_{ij}\, K^{ij} = 0,$$ Hamiltonian constraint

$$\mathcal{M}^i := \nabla_j \big(K^{ij} - \gamma^{ij} K\big) = 0.$$ momentum constraint

A long time of darkness...

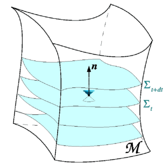


ADM: Richard Arnowitt, Stanley Deser, Charles W. Misner

Figure: 3+1 Orthogonal and Conformal Decomposition of the Einstein Equation and the ADM Formalism for General Relativity, Suat Dengiz. arXiv (2021)

GENERAL RELATIVITY: NUMERICAL ANALYSIS AS A TOOL FOR DISCOVERY

3+1 Einstein equations (ADM form):
[γ: 3-metric in 3+1 decomposition, α, β: lapse & shift (gauge freedom)]



$$\partial_t \gamma_{ij} = -2\,\alpha\,K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i, \qquad \text{evolution of 3-metric}$$

$$\begin{aligned}\partial_t K_{ij} = & -\nabla_i \nabla_j\,\alpha + \alpha\big({}^{(3)}R_{ij} + K\,K_{ij} - 2\,K_{i\ell}\,K_j{}^{\ell}\big)\\ & + \beta^{\ell}\nabla_{\ell} K_{ij} + K_{\ell i}\nabla_j \beta^{\ell} + K_{\ell j}\nabla_i \beta^{\ell},\end{aligned} \qquad \text{evolution of extrinsic curvature (embedding)}$$

$$\mathcal{H} := {}^{(3)}R + K^2 - K_{ij}\,K^{ij} = 0, \qquad \text{Hamiltonian constraint}$$

$$\mathcal{M}^i := \nabla_j\big(K^{ij} - \gamma^{ij} K\big) = 0. \qquad \text{momentum constraint}$$

A long time of darkness...
until The 2005 Breakthrough in Binary Black Hole Mergers by Frans Pretorius
(role of hyperbolicity was recognised). A lot of progress later, contributing to the first detection of gravitational waves in 2015. However,

- little “numerical analysis”,
- next generation of gravitational wave detectors require stability and precision beyond current reach.

Challeges: nonlinear constraints, tensor symmetries, singularity...

Frans Pretorius, Fundamental Physics Breakthrough Prize 2017

Figure: 3+1 Orthogonal and Conformal Decomposition of the Einstein Equation and the ADM Formalism for General Relativity, Suat Dengiz. arXiv (2021)

GENERAL RELATIVITY: NUMERICAL ANALYSIS AS A TOOL FOR DISCOVERY

Einstein–Bianchi system: L.Andersson, V.Moncrief 2024, H.Friedrich 1981

$$E_{ij} = R^{0}{}_{i0j}, \quad B_{ji} = \frac{1}{2} N^{-1} \eta_{ihk} R_{0j}{}^{hk}.$$

Tensor version of Maxwell (linear version):

$$\begin{aligned} B_t + \nabla \times E &= 0, \\ E_t - \nabla \times B &= 0, \\ \nabla \cdot B &= 0, \\ \nabla \cdot E &= 0. \end{aligned}$$

E, B: Transverse-Traceless (TT: symmetric $\mathbb{S}$, trace-free $\mathbb{T}$, divergence-free) tensor fields encoded in BGG conformal complexes

$$0 \longrightarrow C^\infty \xrightarrow{\text{dev hess}} \underbrace{C^\infty \otimes (\mathbb{S} \cap \mathbb{T})}_{E} \xrightarrow{\text{sym curl}} \underbrace{C^\infty \otimes (\mathbb{S} \cap \mathbb{T})}_{B} \xrightarrow{\text{div div}} C^\infty \longrightarrow 0$$

Vincent Quenneville-Bélair, PhD thesis 2015 (U.Minnesota): Finite Element Exterior Calculus formulations

Open: fully encoding tensor symmetries, discretise conformal complexes, nonlinear formulation and constraint-preservation, boundary conditions...

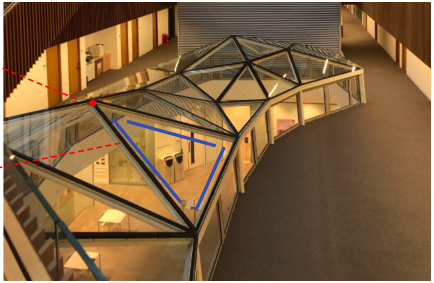
Discrete differential geometry and data sciences: discrete structures v.s. discretisation

Discrete Differential Geometry

Christiansen 2011: Regge calculus = finite elements

- Regge calculus (quantum & numerical gravity) : edge length as metric, angle deficit as curvature
- Regge finite element : piecewise constant symmetric tensor field

$$\text{discrete definitions} \quad \Longrightarrow \quad \text{functions/measures with weak regularity}$$



Finite elements: piecewise functions/measure)

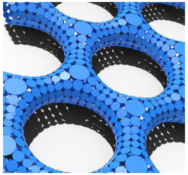
How to define curvature?

metric g discontinuous, $\Gamma \sim g^{-1}(\frac{\partial g}{\partial x} + \frac{\partial g}{\partial x} - \frac{\partial g}{\partial x})$ delta measure, $R \sim \frac{\partial \Gamma}{\partial x} - \frac{\partial \Gamma}{\partial x} + \Gamma\Gamma - \Gamma\Gamma$ not defined!

Further question:

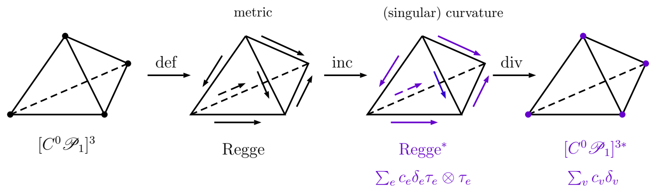
- cohomological techniques for geometric objects (curvature etc.) with ultra weak regularity
  Discrete Geometric Analysis via finite elements and PDEs ?
- finite element approach for other geometric patterns

Consequences: high-order & rigorous & new Discrete Differential Geometry, with applications to advanced materials (origami etc.) , computer graphics , singular structures in universe / GR etc.



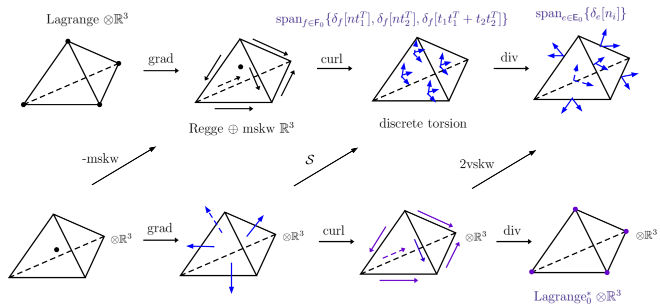
Extend Regge calculus/finite element (S.Christiansen 2011, Regge for curvature)

metric curvature



to Riemann-Cartan geometry (S.Christiansen, KH, L.Ting 2023, extended Regge for curvature + torsion)

coframe (metric) discrete torsion

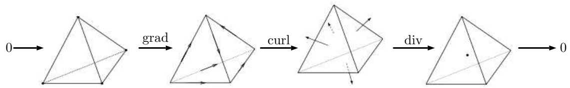


discrete connection curvature

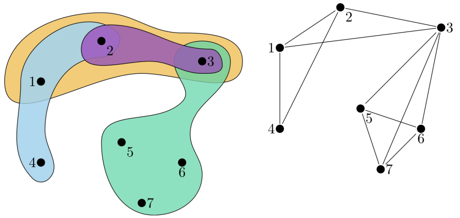
Question: systematic discretisation for generalised continuum with defects, as in Yavari–Goriely?

Topological/geometric data analysis

Canonical finite elements generalise to graphs and networks



Finite element de Rham complex (shown in 3D)



Hypergraphs. Cliques (loops) can exist in any dimension.

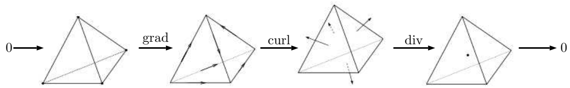
Many applications in Topological Data Analysis (persistent homology), Hodge Laplacian on graphs (ranking, data representation, geometric deep learning...), random graphs and phase transition

Hodge Laplacians on graphs. L. H. Lim, SIAM Review (2020).

What are higher-order networks?. C. Bick, E. Gross, H.A. Harrington, & M.T. Schaub, SIAM Review (2023).

TOPOLOGICAL/GEOMETRIC DATA ANALYSIS

Canonical finite elements generalise to graphs and networks



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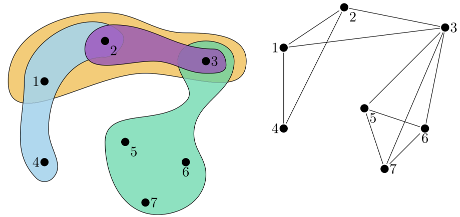
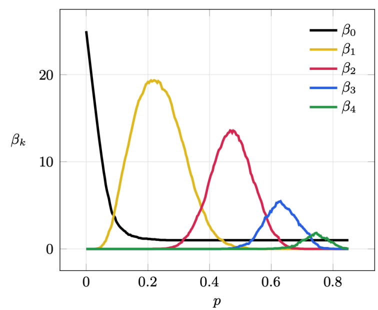


Hypergraphs. Cliques (loops) can exist in any dimension.

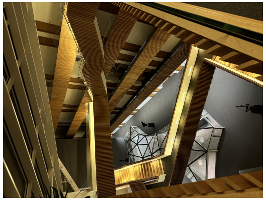
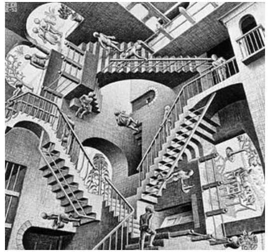
Many applications in Topological Data Analysis (persistent homology), Hodge Laplacian on graphs (ranking, data representation, geometric deep learning...), random graphs and phase transition

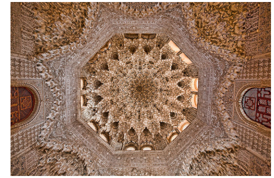
> *I predict a new subject of statistical topology. Rather than count the number of holes, Betti numbers, etc., one will be more interested in the distribution of such objects on non-compact manifolds as one goes out to infinity.*
>
> *– Isadore Singer*



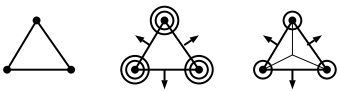
Lázár Bertók, MSc thesis at University of Edinburgh, 2024
Betti number β_k changes with the probability p of a random graph

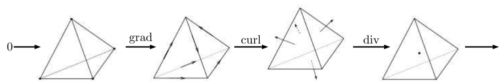
Mathematical elegance lies in patterns , or structures .

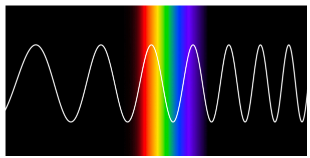
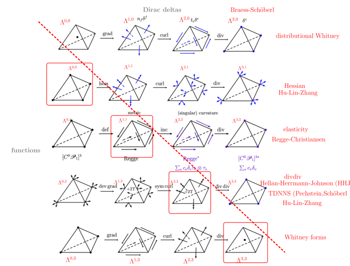





Through structure-aware formulation and structure-preserving discretisation,
achieve computation that we trust.





“If an atom or electron is a basic unit for physicists, his unit is the tetrahedron.”

– Cascading Principles Exhibition, AWB.