Finite elements for symmetric and traceless tensors in three dimensions

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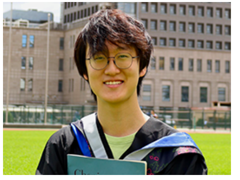

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MOTIVATION

stress, strain tensors, dislocation density, disclination density in continuum mechanics,
metric, curvature (scalar, Ricci, Weyl, Riemann, Cotton...), torsion in differential geometry etc.

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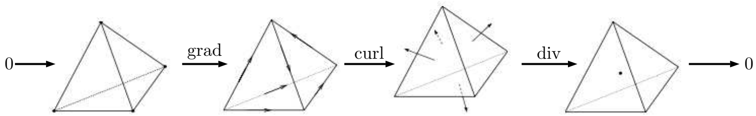
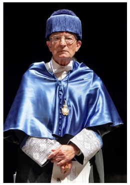
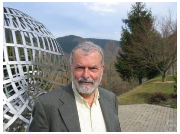
A special case: differential forms (fully skew-symmetric tensors), exterior derivatives



Raviart-Thomas (1977), Nédélec (1980) in numerical analysis
Bossavit (1988): differential forms and complex
Hiptmair (1999), Arnold, Falk, Winther (2006): systematic study, "Finite Element Exterior Calculus"


Pierre-Arnaud Raviart


Jean-Claude Nédélec


Franco Brezzi


Donatella Marini


Jim Douglas

EINSTEIN EQUATIONS

spacetime geometry

matter

$$G_{\alpha\beta} = \frac{8\pi}{c^4} T_{\alpha\beta}$$

Numerically solving the Einstein equations (numerical relativity) has been used to compute templates of gravitational waves and investigate new theories of gravity.

Connection from metric:

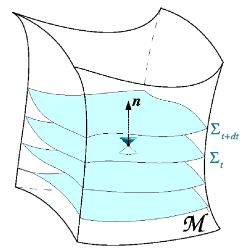
$$\Gamma^k_{ij} = g^{k\ell}\left(\frac{\partial g_{\ell i}}{\partial x^j} + \frac{\partial g_{\ell j}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^\ell}\right),$$

Riemannian tensor from connection:

$$R^\ell{}_{ijk} = \frac{\partial \Gamma^\ell_{ik}}{\partial x^j} - \frac{\partial \Gamma^\ell_{ij}}{\partial x^k} + \Gamma^\ell_{jm}\Gamma^m_{ik} - \Gamma^\ell_{km}\Gamma^m_{ij}.$$

Ricci tensor is the trace of Riemann: $R_{ik} = R^\ell{}_{i\ell k}$;
Einstein tensor is Ricci with modified trace:

$$G_{ik} = R_{ik} - \frac{1}{2} R g_{ik},$$



A major approach: 3+1 (space+time) decomposition

Challeges: nonlinear constraints, tensor symmetries, singularity...

EINSTEIN-BIANCHI FORMULATION

From Bianchi identity:

$$\nabla_\alpha R^\alpha{}_{\beta,\lambda\mu} + \nabla_\mu R_{\lambda\beta} - \nabla_\lambda R_{\mu\beta} = 0.$$

Using the Einstein equations $R_{\alpha\beta} = \rho_{\alpha\beta}$,

$$\nabla_\alpha R^\alpha{}_{\beta,\lambda\mu} = \nabla_\lambda \rho_{\mu\beta} - \nabla_\mu \rho_{\lambda\beta}.$$

Constraint and evolutionary eqns are different components.
Define

$$\boldsymbol{E}_{ij} = R^0{}_{i,0,j}, \quad \boldsymbol{D}_{ij} = \frac{1}{4}\eta_{ihk}\eta_{jlm}R^{hk,lm},$$

$$\boldsymbol{H}_{ij} = \frac{1}{2}N^{-1}\eta_{ihk}R^{hk}{}_{0j}, \quad \boldsymbol{B}_{ji} = \frac{1}{2}N^{-1}\eta_{ihk}R_{0j}{}^{hk}.$$

Now $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{H}$, $\boldsymbol{B}$ satisfy an eqn of Maxwell's type. Linearization around Minkowski:

$$\boldsymbol{B}_t + \nabla \times \boldsymbol{E} = 0,$$
$$\boldsymbol{E}_t - \nabla \times \boldsymbol{B} = 0.$$

$\boldsymbol{E}$, $\boldsymbol{B}$: Traceless-Transverse matrices (symmetric, tracefree, divergence-free), preserved by evolution!

Challenge: encoding symmetries ($\mathbb{S} \cap \mathbb{T}$) and differential structures (divergence-free) in numerics.

- Quenneville-Belair, Vincent. "A new approach to finite element simulations of general relativity." (2015). Thesis with Douglas Arnold. imposing symmetries weakly by Lagrange multipliers

Conformal deformation complex encodes TT tensors

conformal Killing

cott : Cotton-York

stress-like formulation for Stokes
$\sigma :=$ sym grad u

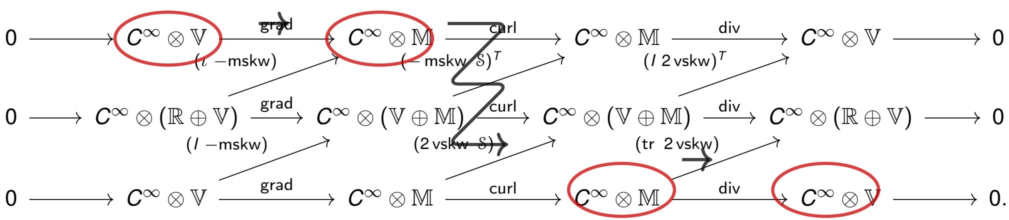
$$0 \longrightarrow C^\infty \otimes \mathbb{V} \xrightarrow{\text{dev def}} C^\infty \otimes (\mathbb{S} \cap \mathbb{T}) \xrightarrow{\text{cott}} C^\infty \otimes (\mathbb{S} \cap \mathbb{T}) \xrightarrow{\text{div}} C^\infty \otimes (\mathbb{V}) \to 0$$

gravitational wave: TT tensor

$\mathbb{S}$: symmetric matrices $\mathbb{T}$: trace-free matrices

dev $w := w - \frac{1}{n}\operatorname{tr}(w)I$,   cott $g :=$ curl S^{-1} curl S^{-1} curl,   div $v := \nabla \cdot v$, $Su := u^T - \operatorname{tr}(u)I$

BGG (Bernstein-Gelfand-Gelfand) point of view: (Arnold, Hu 2021; Čap, Hu 2023)



$\mathbb{M}$: matrix, $\mathbb{V} = \mathbb{R}^3$

Question

Constructing "good" conforming finite element subcomplex of

$$\mathcal{CK} \xrightarrow{\subset} H^1(\Omega;\mathbb{R}^3) \xrightarrow{\text{dev def}} H(\text{cott},\Omega;\mathbb{S}\cap\mathbb{T}) \xrightarrow{\text{cott}} H(\text{div},\Omega;\mathbb{S}\cap\mathbb{T}) \xrightarrow{\text{div}} L^2(\Omega;\mathbb{R}^3) \longrightarrow \mathbf{0},$$

denoted by

$$\mathcal{CK} \xrightarrow{\subset} \boldsymbol{U}_{k+1,h} \xrightarrow{\text{dev def}} \boldsymbol{\Sigma}^{\text{cott}}_{k,h} \xrightarrow{\text{cott}} \boldsymbol{\Sigma}^{\text{div}}_{k-3,h} \xrightarrow{\text{div}} \boldsymbol{V}_{k-4,h} \longrightarrow \mathbf{0}.$$

Sub-problem 1: divergence pair. Construct finite element spaces $\boldsymbol{\Sigma}^{\text{div}}_h$, $\boldsymbol{V}_h$,

$$\cdots \longrightarrow \boldsymbol{\Sigma}^{\text{div}}_h \xrightarrow{\text{div}} \boldsymbol{V}_h \longrightarrow 0,$$

satisfying

$$\text{div}\,\boldsymbol{\Sigma}^{\text{div}}_h = \boldsymbol{V}_h, \qquad \inf_{\boldsymbol{v}\in\boldsymbol{V}_h\setminus\{0\}} \sup_{\boldsymbol{\sigma}\in\boldsymbol{\Sigma}^{\text{div}}_h\setminus\{0\}} \frac{\int_\Omega \text{div}\,\boldsymbol{\sigma}\cdot\boldsymbol{v}}{\|\boldsymbol{\sigma}\|_{H(\text{div},\Omega)}\|\boldsymbol{v}\|_{L^2(\Omega)}} \geq C > 0$$

- Fluid mechanics: $\boldsymbol{\Sigma}^{\text{div}}_h \subset [H^1]^n$ (velocity, vector), $\boldsymbol{V}_h \subset L^2$ (pressure, scalar),
- Elasticity: $\boldsymbol{\Sigma}^{\text{div}}_h \subset H(\text{div};\mathbb{S})$ (stress, sym matrix), $\boldsymbol{V}_h \subset [L^2]^n$ (load, vector),
- General relativity: $\boldsymbol{\Sigma}^{\text{div}}_h \subset H(\text{div};\mathbb{S}\cap\mathbb{T})$ (stress, sym & traceless matrix), $\boldsymbol{V}_h \subset [L^2]^n$ (load, vector)

$$\mathcal{CK} \xrightarrow{\subset} \boldsymbol{U}_{k+1,h} \xrightarrow{\text{dev def}} \boldsymbol{\Sigma}^{\text{cott}}_{k,h} \xrightarrow{\text{cott}} \boldsymbol{\Sigma}^{\text{div}}_{k-3,h} \xrightarrow{\text{div}} \boldsymbol{V}_{k-4,h} \longrightarrow \boldsymbol{0}.$$

Sub-problem 2: $H(\mathrm{cott}; \mathbb{S} \cap \mathbb{T})$-conforming finite elements. conformity conditions from integration by parts

Sub-problem 3: complex, exactness, cohomology. On contractible domains,

$$\ker(\text{cott}, \boldsymbol{\Sigma}^{\text{cott}}_{k,h}) = \text{dev def } \boldsymbol{U}_{k+1,h}$$

$$\ker(\text{div}, \boldsymbol{\Sigma}^{\text{div}}_{k-3,h}) = \text{cott } \boldsymbol{\Sigma}^{\text{cott}}_{k,h},$$

$$\text{div } \boldsymbol{\Sigma}^{\text{div}}_{k-3,h} = \boldsymbol{V}_{k-4,h}$$

SUB-PROBLEM 1: DIVERGENCE PAIR

A classical question in Stokes problem and linear elasticity.

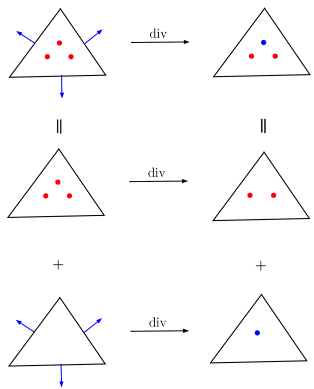
Idea: using bubbles. Thus L^2 pressure is *almost* controlled by *interior part* of $[H^1]^n$ velocity.

$$\cdots \longrightarrow \boldsymbol{\Sigma}^{\mathrm{div}} \xrightarrow{\mathrm{div}} \boldsymbol{V} \longrightarrow 0$$

- div : $[H_0^1]^n \to L^2/\mathbb{R}$ onto, where $\mathbb{R} = \ker(\mathrm{grad})$,
- div : $H_0(\mathrm{div}; \mathbb{S}) \to L^2/\mathcal{RM}$ onto, where $\mathcal{RM} = \ker(\mathrm{sym\,grad})$: infinitesimal rigid body motion
- div : $H_0(\mathrm{div}; \mathbb{S} \cap \mathbb{T}) \to L^2/\mathcal{CK}$ onto, where $\mathcal{CK} = \ker(\mathrm{dev\,sym\,grad})$: conformal Killing fields

Similarly, in finite elements,

$$\begin{array}{ccccccccl}
\cdots & \longrightarrow & \boldsymbol{\Sigma}_h^{\mathrm{div}} & \xrightarrow{\mathrm{div}} & \boldsymbol{V}_h & \longrightarrow & 0 & & \text{full} \\
 & & \| & & \| & & & & \\
\cdots & \longrightarrow & \mathring{\boldsymbol{\Sigma}}_h^{\mathrm{div}} & \xrightarrow{\mathrm{div}} & \boldsymbol{V}_h/\ker(\mathrm{div}^*) & \longrightarrow & 0 & & \text{bubble} \\
 & & + & & + & & & & \\
\cdots & \longrightarrow & \tilde{\boldsymbol{\Sigma}}_h^{\mathrm{div}} & \xrightarrow{\mathrm{div}} & \ker(\mathrm{div}^*) & \longrightarrow & 0 & & \text{skeleton}
\end{array}$$



DIV-BUBBLES: $\mathbb{S}$, $\mathbb{T}$ AND $\mathbb{S} \cap \mathbb{T}$

Theorem 1 (div of symmetric ($\mathbb{S}$) bubbles (Arnold,Awanou,Winther, 2008, Hu & Zhang, 2015))

$$\operatorname{div} \mathbb{B}_k^{\operatorname{div}}(\mathbb{S}) = P_{k-1}(\mathbb{R}^3)/\mathcal{RM},$$

where $\mathbb{B}_k^{\operatorname{div}}(\mathbb{S}) = \{\boldsymbol{\sigma} \in P_k(\mathbb{S}) : \boldsymbol{\sigma}\boldsymbol{n}|_F = \mathbf{0}\}$.

[Arnold,Awanou,Winther, 2008] bubble complex,

[Hu & Zhang] used explicit characterization of bubbles $\mathbb{B}_k^{\operatorname{div}}(\mathbb{S}) = \sum_e \boldsymbol{t}_e\boldsymbol{t}_e^T P_{k-2}(\mathbb{R})$.

Theorem 2 (div of traceless ($\mathbb{T}$) bubbles (Hu & Liang, 2020))

$$\operatorname{div} \mathbb{B}_k^{\operatorname{div}}(\mathbb{T}) = P_{k-1}(\mathbb{R}^3)/\mathcal{RT}.$$

where $\mathbb{B}_k^{\operatorname{div}}(\mathbb{T}) = \{\boldsymbol{\sigma} \in P_k(\mathbb{T}) : \boldsymbol{\sigma}\boldsymbol{n}|_F = \mathbf{0}\}, \quad \mathcal{RT} = \{a\boldsymbol{x} + \boldsymbol{b} : a \in \mathbb{R}, \boldsymbol{b} \in \mathbb{R}^3\} = \ker(\operatorname{dev}\operatorname{grad})$.

Conjecture: $\mathbb{S} \cap \mathbb{T}$

$$\operatorname{div} \mathbb{B}_k^{\operatorname{div}}(\mathbb{S} \cap \mathbb{T}) = P_{k-1}(\mathbb{R}^3)/\mathcal{CK}.$$

where $\mathbb{B}_k^{\operatorname{div}}(\mathbb{S} \cap \mathbb{T}) = \{\boldsymbol{\sigma} \in P_k(\mathbb{S} \cap \mathbb{T}) : \boldsymbol{\sigma}\boldsymbol{n}|_F = \mathbf{0}\}$.

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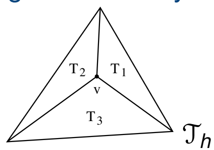
However, the conjecture is **false**!

ALL IS ABOUT *supersmoothness*...

Splines have automatic higher continuity at corners.



$$C^1(\mathcal{T}_h) \implies C^2(v)$$

- Sorokina, T. (2010). Intrinsic supersmoothness of multivariate splines. Numerische Mathematik, 116, 421-434.
- Shekhtman, B., & Sorokina, T. (2015). Intrinsic Supermoothness. Journal of Concrete & Applicable Mathematics, 13.
- Floater, M. S., & Hu, K. (2020). A characterization of supersmoothness of multivariate splines. Advances in Computational Mathematics, 46(5), 70.

Bubbles have higher vanishing properties at corners. e.g., Lagrange bubble $\partial(\lambda_0\lambda_1\lambda_2\lambda_3) = 0$ at vertices

$\boldsymbol{\sigma} \in H(\mathrm{div}; \mathbb{W})$: $\boldsymbol{\sigma} \cdot \boldsymbol{n} = 0$ on faces

$\mathbb{W}$	**Continuity**
$\mathbb{R}^3$	$\boldsymbol{\sigma} = 0$ at vertices
$\mathbb{S}$	$\boldsymbol{\sigma} = 0$ at vertices
$\mathbb{T}$	$\boldsymbol{\sigma} = 0$ at vertices
$\mathbb{S} \cap \mathbb{T}$	$\boldsymbol{\sigma} = \partial\boldsymbol{\sigma} = 0$ at vertices

[Hint: count conditions at a vertex; fewer components in $\mathbb{W} \implies$ more likely higher-order derivatives match.]

ALL IS ABOUT *supersmoothness*...

Define $\operatorname{div}\mathbb{B}_k^{\mathrm{div},(s)}(\mathbb{S}\cap\mathbb{T})$: $\boldsymbol{\sigma}=\partial\boldsymbol{\sigma}=\cdots=\partial^s\boldsymbol{\sigma}=0$ at vertices in addition; similar for $P_{k-1}^{(s-1)}$.

Supersmoothness result: $\mathbb{B}_k^{\mathrm{div}}(\mathbb{S}\cap\mathbb{T})=\mathbb{B}_k^{\mathrm{div},(0)}(\mathbb{S}\cap\mathbb{T})=\mathbb{B}_k^{\mathrm{div},(1)}(\mathbb{S}\cap\mathbb{T})$

Hope:

$$\operatorname{div}\mathbb{B}_k^{\mathrm{div},(s)}(\mathbb{S}\cap\mathbb{T})=P_{k-1}^{(s-1)}(\mathbb{R}^3)/\mathcal{CK}\quad\text{for some } s.$$

Theorem 3

The above **holds for $s=3$, but not for $s=1,2$.**

Sketch of Proof

To count $\dim\mathcal{R}(\operatorname{div})$, we instead count $\mathbb{B}_k^{\mathrm{div},(s)}(\mathbb{S}\cap\mathbb{T})\cap\ker(\operatorname{div})$ through complex of bubbles.

$$\begin{aligned}\dim\mathcal{R}(\operatorname{div})&=\dim\mathbb{B}_k^{\mathrm{div},(s)}(\mathbb{S}\cap\mathbb{T})-\dim\ker(\operatorname{div})\\&=\dim\mathbb{B}_k^{\mathrm{div},(s)}(\mathbb{S}\cap\mathbb{T})-\dim\mathcal{R}(\operatorname{cott})\\&=\dim\mathbb{B}_k^{\mathrm{div},(s)}(\mathbb{S}\cap\mathbb{T})-(\dim\mathbb{B}_k^{\mathrm{cott}}(K;\mathbb{S}\cap\mathbb{T})-\dim\mathcal{R}(\operatorname{dev}\operatorname{def}))\end{aligned}$$

$$\mathcal{CK}\xrightarrow{\subset}\boldsymbol{U}_{k+1,h}\xrightarrow{\operatorname{dev}\operatorname{def}}\boldsymbol{\Sigma}_{k,h}^{\mathrm{cott}}\xrightarrow{\operatorname{cott}}\boldsymbol{\Sigma}_{k-3,h}^{\mathrm{div}}\xrightarrow{\operatorname{div}}\boldsymbol{V}_{k-4,h}\longrightarrow\boldsymbol{0}.$$

What's next

construct first several spaces of the bubble complex.

Lemma 1

Integration by parts for cott *For sufficiently smooth* $\boldsymbol{\sigma}$ *and* $\boldsymbol{\tau}$,

$$(\operatorname{cott}\boldsymbol{\tau},\boldsymbol{\sigma})_K - (\operatorname{cott}\boldsymbol{\sigma},\boldsymbol{\tau})_K = (\operatorname{tr}_1(\boldsymbol{\sigma}), \mathit{\Pi}_F \operatorname{inc}\boldsymbol{\tau}\mathit{\Pi}_F)_{\partial K} - (\operatorname{tr}_3(\boldsymbol{\sigma}), \boldsymbol{n}\times\boldsymbol{\tau}\times\boldsymbol{n})_{\partial K}$$
$$+ (\operatorname{tr}_2(\boldsymbol{\sigma}), 2\operatorname{def}_F(\boldsymbol{n}\cdot\boldsymbol{\tau}\mathit{\Pi}_F) - \mathit{\Pi}_F\partial_n\boldsymbol{\tau}\mathit{\Pi}_F)_{\partial K} + \textit{edge terms},$$

where

$$\operatorname{tr}_1(\boldsymbol{\sigma}) = \operatorname{sym}(\mathit{\Pi}_F\boldsymbol{\sigma}\times\boldsymbol{n}),$$ *similar to* $H(\operatorname{curl})$

$$\operatorname{tr}_2(\boldsymbol{\sigma}) = \operatorname{sym}((2\operatorname{def}_F(\boldsymbol{n}\cdot\boldsymbol{\sigma}\mathit{\Pi}_F) - \mathit{\Pi}_F\partial_n\boldsymbol{\sigma}\mathit{\Pi}_F)\times\boldsymbol{n}),$$ *involving 1st order differential*

$$\operatorname{tr}_3(\boldsymbol{\sigma}) = 2\operatorname{def}_F(\boldsymbol{n}\cdot\operatorname{sym}\operatorname{curl}\boldsymbol{\sigma}\mathit{\Pi}_F) - \mathit{\Pi}_F\partial_n(\operatorname{sym}\operatorname{curl}\boldsymbol{\sigma})\mathit{\Pi}_F.$$ *involving 2nd order differential*

Recall: $\operatorname{cott} := \operatorname{curl}\circ S^{-1}\circ\operatorname{curl}\circ S^{-1}\operatorname{curl}$, where $S\sigma := \sigma^T - \operatorname{tr}(\sigma)I$.

Theorem 4

Let $\boldsymbol{\sigma}$ *be* $\mathbb{S}\cap\mathbb{T}$ *and piecewise polynomials defined on* $\mathcal{T}_h$.

$\boldsymbol{\sigma}\in H(\operatorname{cott},\mathbb{S}\cap\mathbb{T}) \iff$

$$\begin{cases}\operatorname{tr}_1(\boldsymbol{\sigma}), \operatorname{tr}_2(\boldsymbol{\sigma}), \textit{ and } \operatorname{tr}_3(\boldsymbol{\sigma}) \textit{ single-valued on faces}\\ \boldsymbol{\sigma} \textit{ single-valued on edges}\end{cases}$$

Theorem 5.1

The following conformal bubble complexes are exact:

$$\mathbf{0} \to b_K P_{k-3}(K;\mathbb{R}^3) \xrightarrow{\text{dev def}} \mathbb{B}_k^{\text{cott}}(K;\mathbb{S}\cap\mathbb{T}) \xrightarrow{\text{cott}} \mathbb{B}_{k-3}^{\text{div}}(K;\mathbb{S}\cap\mathbb{T}) \cap \ker(\text{div}) \xrightarrow{\text{div}} \mathbf{0}.$$

$$\mathbf{0} \to b_K^2 P_{k-7}(K;\mathbb{R}^3) \xrightarrow{\text{dev def}} b_K \mathbb{B}_{k-4}^{1\,\text{cott}}(K;\mathbb{S}\cap\mathbb{T}) \xrightarrow{\text{cott}} \mathbb{B}_{k-3}^{\text{div}}(K;\mathbb{S}\cap\mathbb{T}) \cap \ker(\text{div}) \xrightarrow{\text{div}} \mathbf{0}.$$

$$\mathbf{0} \to b_K^3 P_{k-11}(K;\mathbb{R}^3) \xrightarrow{\text{dev def}} b_K^2 \mathbb{B}_{k-8}^{2\,\text{cott}}(K;\mathbb{S}\cap\mathbb{T}) \xrightarrow{\text{cott}} \mathbb{B}_{k-3}^{\text{div}}(K;\mathbb{S}\cap\mathbb{T}) \cap \ker(\text{div}) \xrightarrow{\text{div}} \mathbf{0}.$$

where $b_K = \lambda_0\lambda_1\lambda_2\lambda_3$ (scalar bubble),

$$\mathbb{B}_k^{\text{cott}}(K;\mathbb{S}\cap\mathbb{T}) = \{\boldsymbol{\sigma} \in P_k(\mathbb{S}\cap\mathbb{T}) : \text{tr}_1(\boldsymbol{\sigma})|_F = \text{tr}_2(\boldsymbol{\sigma})|_F = \text{tr}_3(\boldsymbol{\sigma})|_F = \mathbf{0}\},$$

$$\mathbb{B}_{k-4}^{1\,\text{cott}}(K;\mathbb{S}\cap\mathbb{T}) = \{\boldsymbol{\sigma} \in P_{k-4}(\mathbb{S}\cap\mathbb{T}) : b_K\boldsymbol{\sigma} \in \mathbb{B}_k^{\text{cott}}(K;\mathbb{S}\cap\mathbb{T})\},$$

$$\mathbb{B}_{k-8}^{2\,\text{cott}}(K;\mathbb{S}\cap\mathbb{T}) = \{\boldsymbol{\sigma} \in P_{k-8}(\mathbb{S}\cap\mathbb{T}) : b_K^2\boldsymbol{\sigma} \in \mathbb{B}_k^{\text{cott}}(K;\mathbb{S}\cap\mathbb{T})\}.$$

Sketch of proof

Using BGG: conformal = elasticity + div div.

Global finite elements: $H(\mathrm{div};\mathbb{S}\cap\mathbb{T}) - L^2(\mathbb{V})$ Pair

Having figured out the bubbles, we obtain global FE spaces.

$\boldsymbol{\Sigma}^{\mathrm{div}}_{k-3,h} \subset H(\mathrm{div},\mathbb{S}\cap\mathbb{T})$ For $k \geq 10$, shape function space $P_{k-3}(K;\mathbb{S}\cap\mathbb{T})$, degrees of freedom

$$\begin{aligned} D^\alpha \boldsymbol{\tau}(\delta), &\quad \forall|\alpha| \leq 3, \quad \forall \delta \in \mathcal{V}(K), \\ \int_e \boldsymbol{\tau} : \boldsymbol{q}, &\quad \forall \boldsymbol{q} \in P_{k-11}(e;\mathbb{S}\cap\mathbb{T}), \quad \forall e \in \mathcal{E}(K), \\ \int_F \boldsymbol{q}\cdot\boldsymbol{\tau}\cdot\boldsymbol{n}, &\quad \forall \boldsymbol{q} \in P^{(1)}_{k-6}(F;\mathbb{R}^3), \quad \forall F \in \mathcal{F}(K), \\ \int_K \boldsymbol{\tau} : \boldsymbol{q}, &\quad \forall \boldsymbol{q} \in \mathbb{B}^{\mathrm{div},(3)}_{k-3}(K;\mathbb{S}\cap\mathbb{T}). \end{aligned}$$

unisolvence, $H(\mathrm{div})$-conformity

$\boldsymbol{V}_{k-4,h} \subset L^2(\mathbb{V})$ shape function space $P_{k-4}(K;\mathbb{R}^3)$, degrees of freedom

$$\begin{aligned} D^\alpha \boldsymbol{v}(\delta), &\quad \forall|\alpha| \leq 2, \quad \forall \delta \in \mathcal{V}(K), \\ \int_K \boldsymbol{v} : \boldsymbol{q}, &\quad \forall \boldsymbol{q} \in P^{(2)}_{k-4}(K;\mathbb{R}^3). \end{aligned}$$

The rest of the complex

- add trace operators terms (from integration by parts) to DoFs to ensure (minimal) conformity,
- bubble complexes tell us what supersmoothness to put,
- to construct 3D FEs, first construct the edge (1D) and face (2D) versions.

  Face/edge traces are 2D/1D finite elements. Finite Element System idea.

Application to $H(\mathrm{cott})$−conforming finite element:

- vertex DoFs : C^6 supersmoothness,
- edge DoFs : $\mathrm{tr}_2(\boldsymbol{\sigma}) \in H(\mathrm{div}_F \, \mathrm{div}_F, \mathbb{S}_F \cap \mathbb{T}_F)$, $\mathrm{tr}_3(\boldsymbol{\sigma}) \in H(\mathrm{rot}_F, \mathbb{S}_F \cap \mathbb{T}_F)$, through *trace diagram*:

$$
\begin{array}{ccccc}
\boldsymbol{u} & \xrightarrow{\mathrm{dev}\,\mathrm{def}} & \boldsymbol{\sigma} & \xrightarrow{\mathrm{cott}} & \boldsymbol{\tau} \\
\downarrow & & \downarrow & & \downarrow \\
\boldsymbol{u}\cdot\boldsymbol{n} & \xrightarrow{-\mathrm{def}_F\,\mathrm{curl}_F} & \mathrm{tr}_2(\boldsymbol{\sigma}) & \xrightarrow{-\mathrm{div}_F\,\mathrm{div}_F} & \boldsymbol{n}\cdot\boldsymbol{\tau}\cdot\boldsymbol{n}
\end{array}
$$

and

$$
\begin{array}{ccccc}
\boldsymbol{u} & \xrightarrow{\mathrm{dev}\,\mathrm{def}} & \boldsymbol{\sigma} & \xrightarrow{\mathrm{cott}} & \boldsymbol{\tau} \\
\downarrow & & \downarrow & & \downarrow \\
\boldsymbol{u}\times\boldsymbol{n} & \xrightarrow{\frac{1}{2}\mathrm{hess}_F\,\mathrm{div}_F} & \mathrm{tr}_3(\boldsymbol{\sigma}) & \xrightarrow{\mathrm{rot}_F} & \boldsymbol{n}\times\boldsymbol{\tau}\cdot\boldsymbol{n}
\end{array}
$$

Trace operators involved in $H(\mathrm{div}_F\,\mathrm{div}_F;\mathbb{S}_F\cap\mathbb{T}_F)$-conforming spaces: for $e\in\mathcal{E}(F)$,

$$\begin{aligned}\mathrm{tr}_{e,1}(\boldsymbol{\sigma}) &:= \boldsymbol{n}_{F,e}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{n}_{F,e},\\ \mathrm{tr}_{e,2}(\boldsymbol{\sigma}) &:= \partial_{\boldsymbol{t}_{F,e}}(\boldsymbol{t}_{F,e}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{n}_{F,e})+\boldsymbol{n}_{F,e}\cdot\mathrm{div}_F\,\boldsymbol{\sigma}\end{aligned}$$

$H(\mathrm{div}_F\,\mathrm{div}_F;\mathbb{S}_F\cap\mathbb{T}_F)$-bubbles with minimal vanishing conditions:

$$\begin{aligned}\mathbb{B}_k^{\mathrm{div}_F\,\mathrm{div}_F}(F;\mathbb{S}_F\cap\mathbb{T}_F)|_F := \{\boldsymbol{\sigma}\in P_k(F;\mathbb{S}_F)|_F : \mathrm{tr}_{e,1}(\boldsymbol{\sigma})|_e=\mathrm{tr}_{e,2}(\boldsymbol{\sigma})|_e=0,\\ \forall e\in\mathcal{E}(F),\boldsymbol{\sigma}(\delta)=\mathbf{0},\forall\delta\in\mathcal{V}(F)\}.\end{aligned}$$

Theorem 5

The following sequence is exact:

$$\mathbf{0}\longrightarrow b_F^2P_{k-4}^{(3)}(F;\mathbb{R})|_F\xrightarrow{\mathrm{def}_F\,\mathrm{curl}_F}\mathbb{B}_k^{\mathrm{div}_F\,\mathrm{div}_F,(5)}(F;\mathbb{S}_F\cap\mathbb{T}_F)|_F\xrightarrow{\mathrm{div}_F\,\mathrm{div}_F}P_{k-2}^{(3)}(F;\mathbb{R})|_F\backslash P_1^+(F;\mathbb{R})|_F\longrightarrow 0,$$

where

$$P_1^+(F;\mathbb{R})|_F := P_1(F;\mathbb{R})|_F\oplus\{(\Pi_F\boldsymbol{x})\cdot(\Pi_F\boldsymbol{x})\}.$$

$H(\mathrm{div}_F\,\mathrm{div}_F, \mathbb{S}_F \cap \mathbb{T}_F)$–CONFORMING ELEMENTS: DOFS

$$
\begin{aligned}
D_F^{\alpha}\boldsymbol{\sigma}(\delta), &\quad \forall 0 \le |\alpha| \le 5, \quad \forall \delta \in \mathcal{V}(F). \quad \text{supersmoothness}\\
\int_e \mathrm{tr}_{e,1}(\boldsymbol{\sigma}) q, &\quad \forall q \in P_{k-12}(e;\mathbb{R}), \quad \forall e \in \mathcal{E}(F).\\
\int_e \mathrm{tr}_{e,2}(\boldsymbol{\sigma}) q, &\quad \forall q \in P_{k-11}(e;\mathbb{R}), \quad \forall e \in \mathcal{E}(F).\\
\int_F \mathrm{div}_F\,\mathrm{div}_F\,\boldsymbol{\sigma} q, &\quad \forall q \in P_{k-2}^{(3)}(F;\mathbb{R})|_F \backslash P_1^{+}(F;\mathbb{R})|_F.\\
\int_F \boldsymbol{\sigma} : \mathrm{def}_F\,\mathrm{curl}_F(b_F^2 q), &\quad \forall q \in P_{k-4}^{(3)}(F;\mathbb{R})|_F.
\end{aligned}
$$

$H(\mathrm{rot}_F, \mathbb{S}_F \cap \mathbb{T}_F)-$conforming elements: DoFs

$$
\begin{aligned}
D_F^{\alpha}\boldsymbol{\sigma}(\delta), &\quad \forall 0 \le |\alpha| \le 4, \quad \forall \delta \in \mathcal{V}(F).\\
\int_e \boldsymbol{\sigma} : \boldsymbol{q}, &\quad \forall \boldsymbol{q} \in P_{k-10}(e; \mathbb{S}_F \cap \mathbb{T}_F), \quad \forall e \in \mathcal{E}(F).\\
\int_e \mathrm{rot}_F\, \boldsymbol{\sigma} \cdot \boldsymbol{q}, &\quad \forall \boldsymbol{q} \in P_{k-9}(e; \Pi_F \mathbb{R}^3), \quad \forall e \in \mathcal{E}(F).\\
\int_F \boldsymbol{\sigma} : \boldsymbol{q}, &\quad \forall q \in P_{k-6}^{(0)}(F; \mathbb{S}_F \cap \mathbb{T}_F)|_F.
\end{aligned}
$$

Traces of traces

Recall that we are constructing the face modes of 3D elements. To get back to 3D, we need edge trace of face traces.

Let $\boldsymbol{\sigma} \in \mathbb{B}_k^{\mathrm{tr}_1}(K; \mathbb{S} \cap \mathbb{T})$ and $\boldsymbol{\sigma}|_e = \mathbf{0}$, $\forall e \in \mathcal{E}(K)$. Then on edge $e \subset F$,

$$\begin{aligned}
\mathrm{tr}_{e,1}\left(\mathrm{tr}_2(\boldsymbol{\sigma})\right) &= -\boldsymbol{t}_e \cdot (\operatorname{sym}\operatorname{curl}\boldsymbol{\sigma}) \cdot \boldsymbol{t}_e, \\
\mathrm{tr}_{e,2}\left(\mathrm{tr}_2(\boldsymbol{\sigma})\right) &= \boldsymbol{n}_{F,e} \cdot \left(2\partial_{\boldsymbol{t}_e}(\operatorname{sym}\operatorname{curl}\boldsymbol{\sigma}) \cdot \boldsymbol{t}_e - \nabla\left(\boldsymbol{t}_e \cdot (\operatorname{sym}\operatorname{curl}\boldsymbol{\sigma}) \cdot \boldsymbol{t}_e\right)\right), \\
\boldsymbol{t}_{F,e} \cdot \mathrm{tr}_3(\boldsymbol{\sigma}) \cdot \boldsymbol{t}_{F,e} &= \boldsymbol{n} \cdot \left(2\partial_{\boldsymbol{t}_e}(\operatorname{sym}\operatorname{curl}\boldsymbol{\sigma}) \cdot \boldsymbol{t}_e - \nabla\left(\boldsymbol{t}_e \cdot (\operatorname{sym}\operatorname{curl}\boldsymbol{\sigma}) \cdot \boldsymbol{t}_e\right)\right), \\
\boldsymbol{n}_{F,e} \cdot \mathrm{tr}_3(\boldsymbol{\sigma}) \cdot \boldsymbol{t}_{F,e} &= -\boldsymbol{t}_e \cdot \nabla \times (\operatorname{sym}\operatorname{curl}\boldsymbol{\sigma}) \cdot \boldsymbol{t}_e - \frac{1}{2}\partial_{\boldsymbol{t}_e}(\boldsymbol{t}_e \cdot \operatorname{div}\boldsymbol{\sigma}).
\end{aligned}$$

Edge DoFs of $H(\mathrm{cot})$ ensures traces of traces are single-valued.
Further, reformulate edge traces to be independent of the face containing the edge.

DoFs of $\Sigma_{k,h}^{\mathrm{cot}}$

$$D^{\alpha}\boldsymbol{\sigma}(\delta), \quad \forall 0 \le |\alpha| \le 6, \quad \forall \delta \in \mathcal{V}(K).$$

$$\int_e \boldsymbol{\sigma} : \boldsymbol{q}, \quad \forall q \in P_{k-14}(e;\mathbb{S}\cap\mathbb{T}), \quad \forall e \in \mathcal{E}(K).$$

$$\int_e \mathrm{tr}_{e,1}^{\mathrm{cott}}(\boldsymbol{\sigma})q, \quad \forall q \in P_{k-13}(e;\mathbb{R}), \quad \forall e \in \mathcal{E}(K).$$

$$\int_e \boldsymbol{n}_{e\pm} \cdot \mathrm{tr}_{e,2}^{\mathrm{cott}}(\boldsymbol{\sigma})q, \quad \forall q \in P_{k-12}(e;\mathbb{R}), \quad \forall e \in \mathcal{E}(K).$$

$$\int_e \mathrm{tr}_{e,3}^{\mathrm{cott}}(\boldsymbol{\sigma})q, \quad \forall q \in P_{k-12}(e;\mathbb{R}), \quad \forall e \in \mathcal{E}(K).$$

$$\int_e \mathrm{cott}\,\boldsymbol{\sigma} : \boldsymbol{q}, \quad \forall \boldsymbol{q} \in P_{k-11}(e;\mathbb{S}\cap\mathbb{T}), \quad \forall e \in \mathcal{E}(K).$$

$$\int_F \mathrm{tr}_1(\boldsymbol{\sigma}) : \boldsymbol{q}, \quad \forall \boldsymbol{q} \in P_{k-3}^{(4)}(F;\mathbb{S}_F\cap\mathbb{T}_F), \quad \forall F \in \mathcal{F}(K).$$

$$\int_F \boldsymbol{n}\cdot \mathrm{cott}\,\boldsymbol{\sigma}\cdot \boldsymbol{n}q, \quad \forall q \in P_{k-6}^{(1)}(F;\mathbb{R}) \cap P_1^{+}(F;\mathbb{R})^{\perp}, \quad \forall F \in \mathcal{F}(K).$$

$$\int_F \mathrm{tr}_2(\boldsymbol{\sigma}) : \mathrm{def}_F\,\mathrm{curl}_F(b_F^2 q), \quad \forall q \in P_{k-5}^{(3)}(F;\mathbb{R}), \quad \forall F \in \mathcal{F}(K).$$

$$\int_F \mathrm{tr}_3(\boldsymbol{\sigma}) : \boldsymbol{q}, \quad \forall \boldsymbol{q} \in P_{k-8}^{(0)}(F;\mathbb{S}_F\cap\mathbb{T}_F), \quad \forall F \in \mathcal{F}(K).$$

$$\int_K \boldsymbol{\sigma} : \boldsymbol{q}, \quad \forall \boldsymbol{q} \in \mathbb{B}_k^{\mathrm{cott},(6)}(K;\mathbb{S}\cap\mathbb{T}).$$

DOFs of $\boldsymbol{U}_{k+1,h}$

$$D^\alpha \boldsymbol{u}(\delta), \quad 0 \le |\alpha| \le 7, \quad \forall \delta \in \mathcal{V}(K). \qquad \textit{supersmoothness}$$

$$\int_e \boldsymbol{u} \cdot \boldsymbol{q}, \quad \forall \boldsymbol{q} \in P_{k-15}(e; \mathbb{R}^3), \quad \forall e \in \mathcal{E}(K).$$

$$\int_e \partial_{\boldsymbol{n}_{e\pm}} (\boldsymbol{u}) \cdot \boldsymbol{q}, \quad \forall \boldsymbol{q} \in P_{k-14}(e; \mathbb{R}^3), \quad \forall e \in \mathcal{E}(K). \qquad \textit{supersmoothness}$$

$$\int_F \boldsymbol{u} \cdot \boldsymbol{q}, \quad \forall \boldsymbol{q} \in P^{(3)}_{k-5}(F; \mathbb{R}^3), \quad \forall F \in \mathcal{F}(K).$$

$$\int_K \boldsymbol{u} \cdot \boldsymbol{q}, \quad \forall \boldsymbol{q} \in P^{(4)}_{k-3}(K; \mathbb{R}^3).$$

$$\mathcal{CK} \xrightarrow{\subset} \boldsymbol{U}_{k+1,h} \xrightarrow{\text{dev def}} \boldsymbol{\Sigma}^{\text{cott}}_{k,h} \xrightarrow{\text{cott}} \boldsymbol{\Sigma}^{\text{div}}_{k-3,h} \xrightarrow{\text{div}} \boldsymbol{V}_{k-4,h} \longrightarrow \boldsymbol{0}.$$

Summary and outlook

Summary

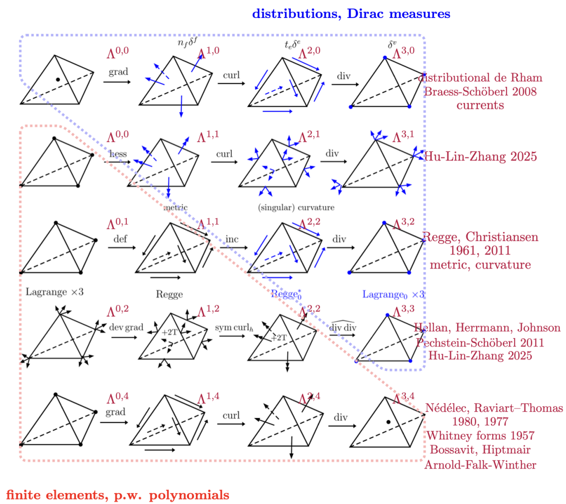
$$\mathcal{CK} \xrightarrow{\subset} \boldsymbol{U}_{k+1,h} \xrightarrow{\operatorname{dev}\operatorname{def}} \boldsymbol{\Sigma}^{\mathrm{cott}}_{k,h} \xrightarrow{\mathrm{cott}} \boldsymbol{\Sigma}^{\mathrm{div}}_{k-3,h} \xrightarrow{\mathrm{div}} \boldsymbol{V}_{k-4,h} \longrightarrow \mathbf{0}.$$

A finite element subcomplex of

$$\mathcal{CK} \xrightarrow{\subset} H^1(\Omega;\mathbb{R}^3) \xrightarrow{\operatorname{dev}\operatorname{def}} H(\mathrm{cott},\Omega;\mathbb{S}\cap\mathbb{T}) \xrightarrow{\mathrm{cott}} H(\mathrm{div},\Omega;\mathbb{S}\cap\mathbb{T}) \xrightarrow{\mathrm{div}} L^2(\Omega;\mathbb{R}^3) \longrightarrow \mathbf{0},$$

For $k \geq 14$: conformity, unisolvence, exactness (on contractible domains).

Questions: Cohomology? Tensor product construction? A more canonical discretization incorporating discrete conformal geometric structure?



Neat pattern of distributional finite elements for symmetric OR trace-free tensors (and BGG complexes), **not** $\mathbb{S}\cap\mathbb{T}$.

KH, Lin 2025, *Finite element form-valued forms: Construction*

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