Towards Computational Topological (Magneto)Hydrodynamics

Relaxation, Dynamo, Finite Element Exterior Calculus

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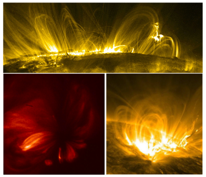
Motivation: structure-preserving discretisation

Fundamental question in plasma physics: given initial data, what does the system evolve to?
heating of solar corona, plasma equilibria (magnetic configurations) etc.



Magneto-friction (simplified MHD) :

$$
\begin{aligned}
\boldsymbol{B}_t - \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) &= 0, \\
\boldsymbol{j} &= \nabla \times \boldsymbol{B}, \\
\boldsymbol{u} &= \tau \boldsymbol{j} \times \boldsymbol{B}.
\end{aligned}
$$

Energy decay

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{B}\|^2 = -\tau\|\boldsymbol{B} \times \boldsymbol{j}\|^2.$$

Helicity conservation

$$\frac{d}{dt}\mathcal{H}_m = 0, \quad \text{with } \mathcal{H}_m := \int \boldsymbol{A}\cdot\boldsymbol{B}\, dx, \quad \boldsymbol{B} = \nabla\times\boldsymbol{A}.$$

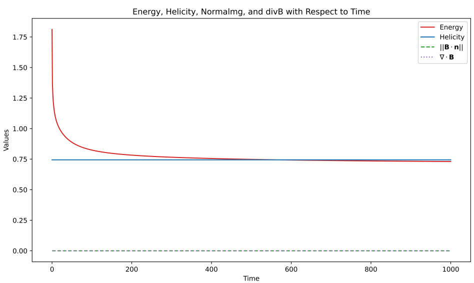


Figure. Helicity-preserving scheme

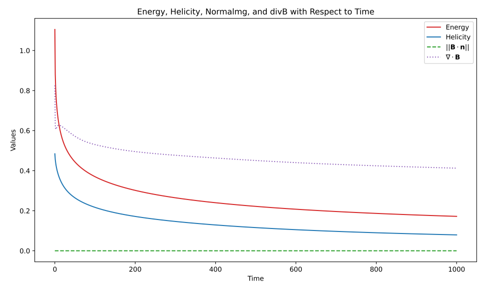
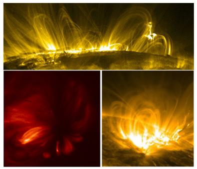
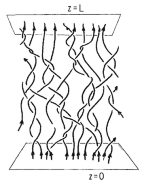


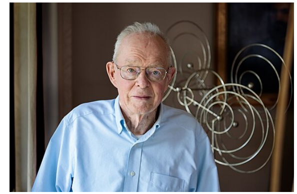
Figure. CG scheme (non-preserving)

- *Topology-preserving discretization for the magneto-frictional equations arising in the Parker conjecture*, M. He, P. E. Farrell, KH, B. Andrews, SISC (2025).

Ideal magnetic relaxation







Eugene Parker

Parker hypothesis (Still Open)

For "almost any initial data", *the magnetic field develops tangential discontinuities (current sheet) during the relaxation to static equilibrium.*

Reliable numerical computation

Computation is used for computing gravitational wave templates , investigating magnetic configurations for fusion devices , designing quantum computing devices etc.

How confident are we in what we compute?

Key: *differential complexes and cohomology* encode fundamental structures in mathematical models.

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How confident are we in what we compute?

Key: *differential complexes and cohomology* encode fundamental structures in mathematical models.

$$\cdots \longrightarrow V^{k-1} \xrightarrow{d^{k-1}} V^k \xrightarrow{d^k} V^{k+1} \longrightarrow \cdots$$

$$0 \longrightarrow C^\infty(\Omega) \xrightarrow{\text{grad}} C^\infty(\Omega;\mathbb{R}^3) \xrightarrow{\text{curl}} C^\infty(\Omega;\mathbb{R}^3) \xrightarrow{\text{div}} C^\infty(\Omega) \longrightarrow 0.$$

$$d^0 := \text{grad}, \quad d^1 := \text{curl}, \quad d^2 := \text{div}\,.$$

- complex property: $d^k \circ d^{k-1} = 0, \;\Rightarrow\; \mathcal{R}(d^{k-1}) \subset \ker(d^k)$,
  $\text{curl} \circ \text{grad} = 0 \Rightarrow \mathcal{R}(\text{grad}) \subset \ker(\text{curl}), \quad \text{div} \circ \text{curl} = 0 \Rightarrow \mathcal{R}(\text{curl}) \subset \ker(\text{div})$
- cohomology: $\mathscr{H}^k := \ker(d^k)/\mathcal{R}(d^{k-1})$,
  $\mathscr{H}^0 := \ker(\text{grad}), \quad \mathscr{H}^1 := \ker(\text{curl})/\mathcal{R}(\text{grad}), \quad \mathscr{H}^2 := \ker(\text{div})/\mathcal{R}(\text{curl})$
- exactness: $\ker(d^k) = \mathcal{R}(d^{k-1})$, i.e., $d^k u = 0 \;\Rightarrow\; u = d^{k-1} v$
  $\text{curl}\, u = 0 \Rightarrow u = \text{grad}\,\phi, \quad \text{div}\, v = 0 \Rightarrow v = \text{curl}\,\psi.$

Outline

Finite element exterior calculus & Relaxation

Magnetohydrodynamics (MHD): macroscopic description of plasma, an incompressible model

$$
\begin{aligned}
\partial_t \boldsymbol{u} - \boldsymbol{u} \times (\nabla \times \boldsymbol{u}) - R_e^{-1} \Delta \boldsymbol{u} - s\boldsymbol{j} \times \boldsymbol{B} + \nabla P &= \boldsymbol{f} && \text{momentum equation,}\\
\boldsymbol{j} - \nabla \times \boldsymbol{B} &= \boldsymbol{0} && \text{Ampere's law,}\\
\partial_t \boldsymbol{B} + \nabla \times \boldsymbol{E} &= \boldsymbol{0} && \text{Faraday's law,}\\
R_m^{-1} \boldsymbol{j} - (\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}) &= \boldsymbol{0} && \text{Ohm's law,}\\
\nabla \cdot \boldsymbol{B} &= 0 && \text{Gauss law,}\\
\nabla \cdot \boldsymbol{u} &= 0,
\end{aligned}
$$

initial conditions $\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}), \quad \boldsymbol{B}(\boldsymbol{x}, 0) = \boldsymbol{B}_0(\boldsymbol{x})$,

boundary conditions on $\partial\Omega$: $\boldsymbol{u} = \boldsymbol{0}, \quad \boldsymbol{B} \cdot \boldsymbol{n} = 0, \quad \boldsymbol{E} \times \boldsymbol{n} = \boldsymbol{0}$.

Three nonlinear terms:

fluid advection $-\boldsymbol{u} \times (\nabla \times \boldsymbol{u})$ (in the vorticity form)
Lorentz force $-s\boldsymbol{j} \times \boldsymbol{B}$
magnetic advection $-\nabla \times (\boldsymbol{u} \times \boldsymbol{B})$

For relaxation, we are interested in zero magnetic diffusion, nonzero fluid diffusion ($R_m = \infty$, $R_e < \infty$).

Energy structures of MHD

Energy dissipation or conservation:

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}\|_0^2 + \frac{S}{2}\frac{d}{dt}\|\boldsymbol{B}\|_0^2 + R_e^{-1}\|\nabla \boldsymbol{u}\|_0^2 + SR_m^{-1}\|\boldsymbol{j}\|_0^2 = (\boldsymbol{f}, \boldsymbol{u}),$$

and hence

$$\max_{0\le t\le T}\left(\|\boldsymbol{u}\|_0^2 + S\|\boldsymbol{B}\|_0^2\right) + R_e^{-1}\int_0^T \|\nabla \boldsymbol{u}\|_0^2 \,\mathrm{d}\tau + 2SR_m^{-1}\int_0^T \|\boldsymbol{j}\|_0^2 \,\mathrm{d}\tau$$
$$\le \|\boldsymbol{u}_0\|_0^2 + S\|\boldsymbol{B}_0\|_0^2 + R_e\int_0^T \|\boldsymbol{f}\|_{-1}^2 \,\mathrm{d}\tau.$$

With $\boldsymbol{f} = 0$, $R_m^{-1} = 0$, total energy is non-increasing. However, some key information is not clear:

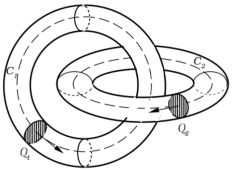
- whether the total energy decays to zero?
- how does total energy split into the fluid part $(\|\boldsymbol{u}\|^2)$ $+$ magnetic part $(S\|\boldsymbol{B}\|^2)$?

Helicity: fine structures

Magnetic helicity: for magnetic potential $\boldsymbol{A}$ satisfying $\nabla \times \boldsymbol{A} = \boldsymbol{B}$,

magnetic helicity $\mathcal{H}_m := \int_\Omega \boldsymbol{A} \cdot \boldsymbol{B}\, dx$

- Idea started from Helmholtz & Kelvin.
  MHD: Woltjer's invariant, ideal fluid: Moffatt (giving the name).
- characterizing linking/knottedness of $\boldsymbol{B}$.
  Example: $\mathcal{H}_\xi = 2I(C_1, C_2)Q_1 \cdot Q_2$, where I is the Gauss linking number (topological quantity, =1 in the figure below).



Arnold, Khesin, *Topological methods in hydrodynamics*, 1999

Helicity = averaging asymptotic linking number (continuum version of linked tubes) (V.I. Arnold)

Cross helicity:

cross helicity $\mathcal{H}_c := \int_\Omega \boldsymbol{u} \cdot \boldsymbol{B}\, dx$

linking of vorticity and magnetic fields

A topological mechanism

Arnold inequality (V.I. Arnold 1974): helicity provides lower bound for energy

$$\left| \int \boldsymbol{A} \cdot \boldsymbol{B} \, dx \right| \leq C \int |\boldsymbol{B}|^2 \, dx$$

Vladimir I. Arnold

Proof. Cauchy-Schwarz $|\int \boldsymbol{A} \cdot \boldsymbol{B} \, dx| \leq \|\boldsymbol{A}\|_{L^2} \|\boldsymbol{B}\|_{L^2}$ + Poincaré inequality $\|\boldsymbol{A}\|_{L^2} \leq C \|\nabla \times \boldsymbol{A}\|_{L^2}$.

Differential form point of view: $\boldsymbol{A}$: 1-form, $\boldsymbol{B}$: 2-form

$$\int \boldsymbol{A} \wedge \boldsymbol{B} \quad \leq C \quad \int \boldsymbol{B} \wedge * \boldsymbol{B}$$

Helicity, Topology | *Energy, Geometry*

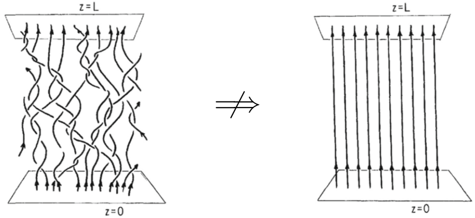


Fig: Pontin, Hornig, Living Rev. Sol. Phys. 2020.

knots are topological barriers that prevent energy from dissipation

Magnetic and cross helicity are conservative for ideal MHD ($R_e^{-1} = R_m^{-1} = 0$).

$$\frac{d}{dt}\int \boldsymbol{A}\cdot\boldsymbol{B}\,dx = 0, \quad \frac{d}{dt}\int \boldsymbol{u}\cdot\boldsymbol{B}\,dx = 0.$$

Proof Advection of magnetic fields:

$$\boldsymbol{B}_t - \nabla\times(\boldsymbol{u}\times\boldsymbol{B}) = 0.$$

Then

$$\frac{d}{dt}\int \boldsymbol{A}\cdot\boldsymbol{B} = 2\frac{d}{dt}\int \boldsymbol{A}\cdot\nabla\times(\boldsymbol{u}\times\boldsymbol{B}) \overset{*}{=} 2\int \nabla\times\boldsymbol{A}\cdot(\boldsymbol{u}\times\boldsymbol{B}) = 2\int \boldsymbol{u}\cdot(\boldsymbol{B}\times\boldsymbol{B}) = 0.$$

$*$: integral by parts with vanishing boundary conditions.

Proof does not depend on $\boldsymbol{u}$. Magnetic helicity conserved even with fluid diffusion.

Consequences: consider a system with fluid diffusion ($R_e < \infty$), without magnetic diffusion ($R_m = \infty$). Energy may decay (due to fluid diffusion), but has a lower bound (by magnetic helicity, which remains constant). So topologically nontrivial initial data cannot evolve to a trivial stationary state. This provides a topological constraint for ideal magnetic relaxation.

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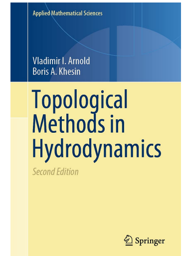
But numerical computation may lose this topological mechanism due to discretization errors (therefore leading to **wrong solutions**)!

Towards *Computational Topological Hydrodynamics*

A subject back to Kelvin, Helmholtz, and more recently by Arnold, Khesin, Moffatt, Sullivan...
limited applications due to lack of topology-preserving algorithms



Lord Kelvin | von Helmholtz | Vladimir Arnold | Boris Khesin | Keith Moffatt | Dennis Sullivan

> *Direct computational assessment of **Parker's hypothesis** brings a number of challenges. Foremost among these is the requirement to **precisely maintain the magnetic topology during the simulated evolution**, i.e., precisely maintain the magnetic field line mapping between the two line-tied boundaries. . . . In the following sections, two methods are described which seek to mitigate against these difficulties. However, in all cases the **representation of current singularities remains problematic**. . .* —

The Parker problem: existence of smooth force-free fields and coronal heating, Pontin, Hornig, Living Rev. Sol. Phys. 2020.

Structure-preserving MHD: literature

Existing numerical methods for magnetic relaxation: Lagrange method, issues with mesh deformation

- *Mimetic methods for Lagrangian relaxation of magnetic fields*, S.Candelaresi, D.Pontin, G.Hornig, SIAM Journal on Scientific Computing (2014).

Structure-preserving discretization for MHD:

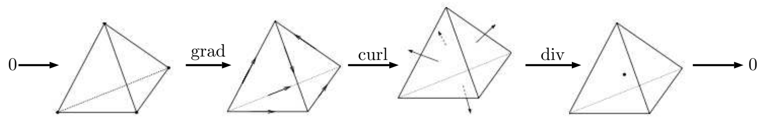
- energy conservation: e.g., Armero, Simo 1996
- $\nabla \cdot \boldsymbol{B} = 0$: e.g., Brackbill, Barnes 1980
- helicity conservation: less attention, Liu,Wang 2004 (axisymmetric MHD flow, finite difference methods); Kraus,Maj 2017 (DEC, variational integrator), Sullivan 2018 ('Lattice hydrodynamics').

Helicity-preserving finite element for NS:
Rebholz 2007; Zhang, Palha, Gerritsma, Rebholz 2022 (dual field approach).

Helicity-preserving finite element for MHD:
KH, Lee, Xu 2021; Gawlik, Gay-Balmaz 2022; Laakmann, KH, Farrell 2023 (Hall MHD), Zhang, Palha, Brugnoli, Toshniwal, Gerritsma 2024.

The numerics below are based on the projection approach (Rebholz 2007, KH, Lee, Xu 2021).

Canonical finite elements for the de Rham complex



Raviart-Thomas (1977), Nédélec (1980) in numerical analysis

"The main advantage of these finite elements is the possibility of approximating Maxwell's equations while exactly verifying one of the physical law." – J.C. Nédélec, Mixed Finitc Elements in $\mathbb{R}^3$ (1980)

Bossavit (1988): differential forms and complex

"A rationale for the use of these special 'mixed' clements can be obtained if one expresses basic equations in terms of differential forms, instead of vector fields. ... Whitney forms were described in 1957, long before the use of finite elements."
– A. Bossavit, Whitney forms: a class of finite elements for three-dimensional computations in electromagnetism (1988)

Hiptmair (1999), Arnold, Falk, Winther (2006): systematic study, "Finite Element Exterior Calculus"

Finite element exterior calculus (FEEC): structure-preserving FEM
Discrete exterior calculus (DEC): defining spaces and operators on primal and dual meshes
Topological data analysis (TDA): cohomology and Hodge-Laplacian on graphs

Lim, Lek-Heng. "Hodge Laplacians on graphs." SIAM Review 62.3 (2020).

Why complexes matter?

Example: Gauss law in Maxwell equations. $\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E} = 0 \Longrightarrow \partial_t (\nabla \cdot \boldsymbol{B}) = 0$.

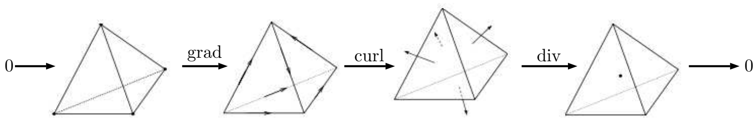
Typical Galerkin schemes: Find $\boldsymbol{E}_h \in Y_h,\ \boldsymbol{B}_h \in Z_h$, s.t.

$$(\partial_t \boldsymbol{B}_h, \boldsymbol{C}_h) + (\nabla \times \boldsymbol{E}_h, \boldsymbol{C}_h) = 0, \quad \forall \boldsymbol{C}_h \in Z_h.$$

Consequence: $\partial_t \boldsymbol{B}_h + \mathbb{P}\nabla \times \boldsymbol{E}_h = 0$, hence $\partial_t(\nabla \cdot \boldsymbol{B}_h) = -\nabla \cdot \mathbb{P}\nabla \times \boldsymbol{E}_h$. Non-zero in general, up to discretization errors.

$\mathbb{P}$: L^2 projection from $\nabla \times Y_h$ to Z_h. $\mathbb{P} = I$ (hence $\partial_t(\nabla \cdot \boldsymbol{B}_h) = 0$) only if $\nabla \times Y_h \subset Z_h$.

Relations like $\nabla \times Y_h \subset Z_h$ are encoded in differential complexes (homological algebra).



Using finite elements in complexes leads to constraint-preservation.

Discretization for MHD

Choice of spaces: all spaces in a finite element de Rham complex

- To preserve $\nabla \cdot \boldsymbol{B} = 0$, discretize $\boldsymbol{B} \in H_0^h(\text{div})$, $\boldsymbol{E} \in H_0^h(\text{curl})$. Complex: curl $H_0^h(\text{curl}) \subset H_0^h(\text{div})$!
- key cancellation for the magnetic helicity : on the continuous level,

$$\int \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) \cdot \boldsymbol{A} = \int (\boldsymbol{u} \times \boldsymbol{B}) \cdot \boldsymbol{B} = 0.$$

  On the discrete level, a natural mixed scheme yields

$$\int \nabla \times \mathbb{Q}_h^{\text{curl}}(\boldsymbol{u} \times \boldsymbol{B}) \cdot \boldsymbol{A} = \int \mathbb{Q}_h^{\text{curl}}(\boldsymbol{u} \times \boldsymbol{B}) \cdot \nabla \times \boldsymbol{A} = \int (\boldsymbol{u} \times \boldsymbol{B}) \cdot \mathbb{Q}_h^{\text{curl}} \boldsymbol{B} \neq 0,$$

  $\mathbb{Q}_h^{\text{curl}}$: L^2 projection to $H_0^h(\text{curl})$.
  Fix: introduce $\boldsymbol{H} = \mathbb{Q}_h^{\text{curl}} \boldsymbol{B}$, use $\nabla \times \mathbb{Q}_h^{\text{curl}}(\boldsymbol{u} \times \boldsymbol{H})$ in the scheme.

$$\int (\boldsymbol{u} \times \mathbb{Q}_h^{\text{curl}} \boldsymbol{B}) \cdot \mathbb{Q}_h^{\text{curl}} \boldsymbol{B} = 0.$$

- cross helicity : similar. Introduce $\boldsymbol{\omega} := \mathbb{Q}_h^{\text{curl}} \nabla \times \boldsymbol{u}$.
- Any time stepping that preserves quadratic invariants.

Numerical scheme (magneto-friction)

Apply the same idea of choosing finite elements in a de Rham complex and adding projections :

Find $(\boldsymbol{B}, \boldsymbol{E}, \boldsymbol{H}, \boldsymbol{j}, \boldsymbol{u}) \in H^h(\mathrm{div}) \times H^h(\mathrm{curl}) \times H^h(\mathrm{curl}) \times H^h(\mathrm{curl}) \times H^h(\mathrm{div})$, such that for any $(\hat{\boldsymbol{B}}, \hat{\boldsymbol{E}}, \hat{\boldsymbol{H}}, \hat{\boldsymbol{j}}, \hat{\boldsymbol{u}})$ in the same space,

$$
\begin{aligned}
(\boldsymbol{B}_t, \hat{\boldsymbol{B}}) + (\nabla \times \boldsymbol{E}, \hat{\boldsymbol{B}}) &= 0, \\
(\boldsymbol{E}, \hat{\boldsymbol{E}}) &= -(\boldsymbol{u} \times \boldsymbol{H}, \hat{\boldsymbol{E}}), \\
(\boldsymbol{u}, \hat{\boldsymbol{v}}) &= \tau(\boldsymbol{j} \times \boldsymbol{H}, \hat{\boldsymbol{v}}), \\
(\boldsymbol{j}, \hat{\boldsymbol{j}}) &= (\boldsymbol{B}, \nabla \times \hat{\boldsymbol{j}}), \\
(\boldsymbol{H}, \hat{\boldsymbol{H}}) &= (\boldsymbol{B}, \hat{\boldsymbol{H}}).
\end{aligned}
\qquad
\begin{aligned}
\boldsymbol{B}_t + \nabla \times \boldsymbol{E} &= 0, \\
\boldsymbol{E} &= -\mathbb{P}(\boldsymbol{u} \times \boldsymbol{H}), \\
\boldsymbol{u} &= \tau\mathbb{Q}(\boldsymbol{j} \times \boldsymbol{H}), \\
\boldsymbol{j} &= \nabla_h \times \boldsymbol{B}, \\
\boldsymbol{H} &= \mathbb{P}\boldsymbol{B}.
\end{aligned}
$$

Energy law

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{B}\|^2 = -\tau\|\mathbb{Q}(\boldsymbol{H} \times \boldsymbol{j})\|^2.$$

Helicity conservation

$$\frac{d}{dt}\int \boldsymbol{A} \cdot \boldsymbol{B} = 0.$$

Numerical test: Hopf fibration

$$\boldsymbol{B}_0 = \frac{4\sqrt{a}}{\pi(1+r^2)^3}(2y(y-xz), -2(x+yz), (-1+x^2+y^2-z^2))$$

Every single field line of this field is a perfect circle, and every single field line is linked with every other one.

c.f. Smiet, C.B., Candelaresi, S. and Bouwmeester, D., 2017. Ideal relaxation of the Hopf fibration. Physics of Plasmas, 24(7).

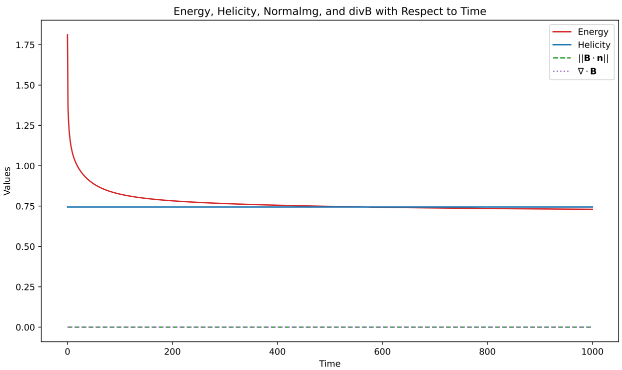


Figure. Helicity-preserving scheme

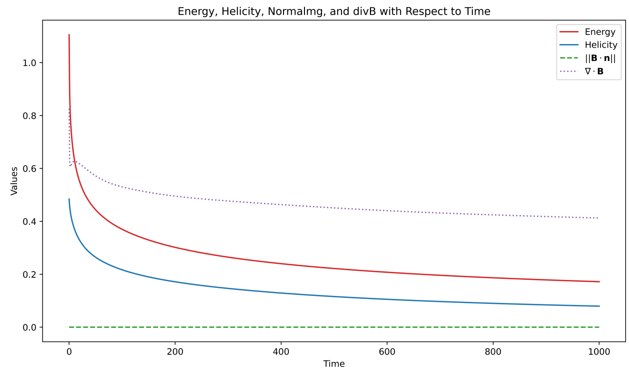
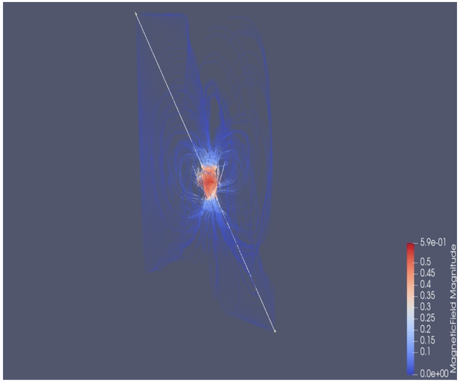


Figure. CG scheme (non-preserving)

$\tau = 10$, $dt = 1$ and $T = 1000$.


5.9e-01
0.5
0.45
0.4
0.35
0.3
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0.2
0.15
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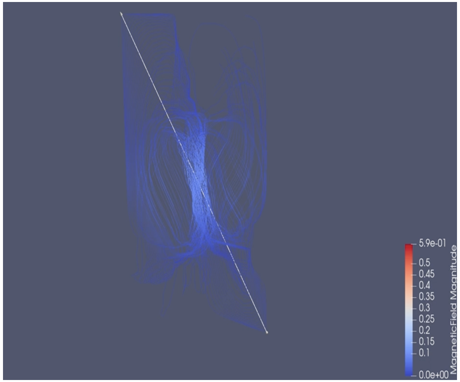
0.0e+00
MagneticField Magnitude


5.9e-01
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.0e+00
MagneticField Magnitude

[show video]

Full MHD (KH, Lee, Xu 2021)

Find $(\boldsymbol{u}, \boldsymbol{\omega}, \boldsymbol{j}, \boldsymbol{E}, \boldsymbol{H}, \boldsymbol{B}, p) \in [H_0^h(\text{curl}, \Omega)]^5 \times H_0^h(\text{div}, \Omega) \times H_0^h(\mathrm{grad})$ such that

$$
\begin{aligned}
(D_t \boldsymbol{u}, \boldsymbol{v}) - (\boldsymbol{u} \times \boldsymbol{\omega}, \boldsymbol{v}) + (\nabla p, \boldsymbol{v}) - S(\boldsymbol{j} \times \boldsymbol{H}, \boldsymbol{v}) &= (\boldsymbol{f}, \boldsymbol{v}), && \text{(1a)} \\
(\boldsymbol{\omega}, \boldsymbol{\mu}) - (\nabla \times \boldsymbol{u}, \boldsymbol{\mu}) &= 0, && \text{(1b)} \\
(\boldsymbol{u}, \nabla q) &= 0, && \text{(1c)} \\
(D_t \boldsymbol{B}, \boldsymbol{C}) + (\nabla \times \boldsymbol{E}, \boldsymbol{C}) &= 0, && \text{(1d)} \\
(\boldsymbol{j}, \boldsymbol{k}) - (\boldsymbol{B}, \nabla \times \boldsymbol{k}) &= 0, && \text{(1e)} \\
(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{H}, \boldsymbol{G}) &= 0, && \text{(1f)} \\
(\boldsymbol{B}, \boldsymbol{F}) - (\boldsymbol{H}, \boldsymbol{F}) &= 0, && \text{(1g)}
\end{aligned}
$$

where $D_t \boldsymbol{u} = (\boldsymbol{u}^{new} - \boldsymbol{u}^{old})/\Delta t$, $D_t \boldsymbol{B} = (\boldsymbol{B}^{new} - \boldsymbol{B}^{old})/\Delta t$ and other variables are average of new and old values (*time stepping: implicit mid-point*).

$$
\begin{aligned}
\boldsymbol{E} &= -\mathbb{Q}_h^{\text{curl}}(\boldsymbol{u} \times \boldsymbol{H}), \\
\boldsymbol{\omega} &= \mathbb{Q}_h^{\text{curl}}(\nabla \times \boldsymbol{u}) \\
\boldsymbol{j} &= \nabla_h \times \boldsymbol{B}, \quad \boldsymbol{H} = \mathbb{Q}_h^{\text{curl}} \boldsymbol{B}.
\end{aligned}
$$

Convergence

Algorithms converge well for *smooth true solutions*.

Theorem 1 (L. Beirão da Veiga, KH, L. Mascotto 2024[1])

Consider sequences $\{\mathcal{T}_h\}$ *of shape-regular, quasi-uniform meshes. Let the true solution be sufficiently smooth. Then, there exists a positive constant* C *independent of* h *such that, for all* t *in* $(0, T]$,

$$\|\mathbf{e}_\mathsf{h}^\mathbf{u}(t)\|^2 + \|\mathbf{e}_\mathsf{h}^\mathbf{B}(t)\|^2 + \int_0^t \|\operatorname{curl}\mathbf{e}_\mathsf{h}^\mathbf{u}(s)\|^2 \,\mathrm{ds} + \int_0^t \|\mathbf{e}_\mathsf{h}^\mathbf{j}(s)\|^2 \,\mathrm{ds} \le C(\|\mathbf{e}_\mathsf{h}^\mathbf{u}(0)\|^2 + \|\mathbf{e}_\mathsf{h}^\mathbf{B}(0)\|^2 + h^{2(k+1)}).$$

The constant C *includes regularity terms of the numerical solution, the shape-regularity parameter of the mesh, and the polynomial degree* k.

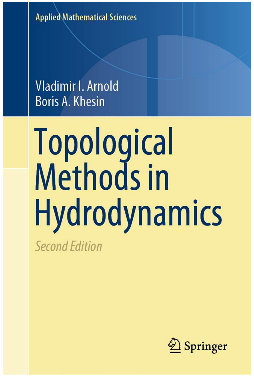
Further question: What if the true solution is **nonsmooth**?

Onsager's conjecture; energy/helicity conservation may fail. But most FE preserves energy by definition.
Where is the boundary of structure-preservation?

[1] L. Beirão da Veiga, KH, L. Mascotto, *Convergence analysis of a helicity-preserving finite element discretisation for an incompressible magnetohydrodynamics system*, arXiv (2024)

Towards *Computational Topological Hydrodynamics*

Dynamo theory, another example:



mechanism of generation of magnetic fields in astrophysical objects
(e.g., change of magnetic fields of stars and planets)

Fast dynamo: exponential growth of magnetic field $\boldsymbol{B}$
First eigenvalue of magnetic advection-diffusion (given $\boldsymbol{u}$)

$$-\nabla \times (\boldsymbol{u} \times \boldsymbol{B}) - R_m^{-1} \nabla \times \nabla \times \boldsymbol{B} = \lambda \boldsymbol{B}.$$

> Does there exist a divergence-free field $\boldsymbol{u}$ on a manifold that is a fast kinematic dynamo?

V.I.Arnold, E.I.Korkina 1983 computation: 'Galerkin methods', magnetic Reynolds number $R_m \leq 19$.

Are there spurious solutions like in Maxwell equations?

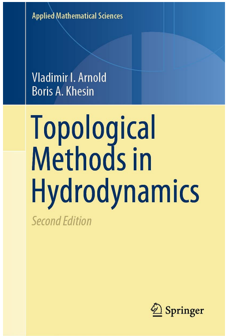
Dynamo

V.I.Arnold, E.I.Korkina 1983 computation: ‘Galerkin methods’, magnetic Reynolds number $R_m \leq 19$.

Are there spurious solutions like in Maxwell equations?

> *... It is* **still unknown** *whether this field (ABC flow) is a fast kinematic dynamo, e.g., whether an exponentially growing mode of B survives as* $R_m \to \infty$.
>
> ...
>
> *Numerically, the kinematic fast dynamo problem is the first eigenvalue problem for matrices of the order of many million, even for reasonable Reynolds numbers (of the order of hundreds). The physically meaningful magnetic Reynolds numbers* R_m *are of order of magnitude* 10^8. **The corresponding matrices are (and will remain) beyond the reach of any computer.**

— *Topological Methods in Hydrodynamics*, V.I.Arnold, B.A.Khesin 2021.



Dynamo

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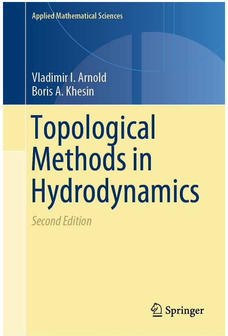
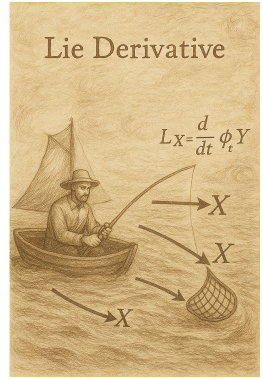


Is this true?

MHD via Differential Forms

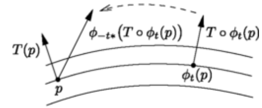
$$\underbrace{-\nabla\times(\boldsymbol{u}\times\boldsymbol{B})}_{\substack{\text{Lie derivative}\\ \text{advection}}}\underbrace{-R_m^{-1}\nabla\times\nabla\times\boldsymbol{B}}_{\substack{\text{Hodge Laplacian}\\ \text{diffusion}}}=\lambda\boldsymbol{B}$$



- Diffusion: Hodge Laplacian . $\Delta_{\mathrm{HL}} := d\delta + \delta d$ (diffusion)
- Lie Derivative: For vector field u on manifold $\mathcal{M}$. For a k-form ω,

$$\mathsf{L}_u\omega = \lim_{\tau\to 0}\frac{\Phi_\tau^*\omega - \omega}{\tau}$$

where flow $\Phi(t,x)$ satisfies $\partial_t\Phi = u(\Phi,t)$, $\Phi(0,x)=x$.



- Cartan's Magic Formula: For vector field β:

$$\mathsf{L}_\beta^k = d^{k-1}i_\beta^k + i_\beta^{k+1}d^k$$

where $i_\beta^k : \Lambda^k \to \Lambda^{k-1}$ is contraction.

' Fisherman derivative ': sitting on boat, differentiating along the flow

$$0 \longleftarrow C^\infty(\Omega) \xleftarrow{\cdot\beta} C^\infty(\Omega;\mathbb{R}^3) \xleftarrow{\times\beta} C^\infty(\Omega;\mathbb{R}^3) \xleftarrow{\otimes\beta} C^\infty(\Omega) \longleftarrow 0.$$

$$\underbrace{\mathsf{L}_u w}_{\substack{\text{Lie derivative}\\ \text{advection}}} + \underbrace{\Delta_{HL} w}_{\substack{\text{Hodge Laplacian}\\ \text{diffusion}}}$$

Numerical application: (semi-)Lagrange methods for MHD Heumann,Hiptmair,Xu 2009

Advection-diffusion of differential forms: in coordinates

$$\mathsf{L}_\beta w + \Delta_{HL} w = f.$$

$$0 \rightleftarrows C^\infty(\Omega) \underset{-\operatorname{div}}{\overset{\operatorname{grad}}{\rightleftarrows}} C^\infty(\Omega;\mathbb{R}^3) \underset{\operatorname{curl}}{\overset{\operatorname{curl}}{\rightleftarrows}} C^\infty(\Omega;\mathbb{R}^3) \underset{-\operatorname{grad}}{\overset{\operatorname{div}}{\rightleftarrows}} C^\infty(\Omega) \rightleftarrows 0.$$

$$0 \longleftarrow C^\infty(\Omega) \xleftarrow{\cdot\beta} C^\infty(\Omega;\mathbb{R}^3) \xleftarrow{\times\beta} C^\infty(\Omega;\mathbb{R}^3) \xleftarrow{\otimes\beta} C^\infty(\Omega) \longleftarrow 0.$$

Advection-diffusion of differential forms: in coordinates

$$\mathrm{L}_\beta w + \varDelta_{HL} w = f.$$

$$0 \rightleftarrows C^\infty(\varOmega) \underset{-\operatorname{div}}{\overset{\operatorname{grad}}{\rightleftarrows}} C^\infty(\varOmega;\mathbb{R}^3) \qquad C^\infty(\varOmega;\mathbb{R}^3) \qquad C^\infty(\varOmega) \qquad 0$$

$$0 \longleftarrow C^\infty(\varOmega) \xleftarrow{\cdot\beta} C^\infty(\varOmega;\mathbb{R}^3) \qquad C^\infty(\varOmega;\mathbb{R}^3) \qquad C^\infty(\varOmega) \qquad 0$$

$$(d^{k-1} i^k_\beta \quad + \quad i^{k+1}_\beta d^k) w \quad + \quad (d^{k-1} d^*_{k-1} \quad + \quad d^*_k d^k) w \quad = \quad f$$

$$\beta \cdot \nabla w \qquad - \qquad \operatorname{div}\operatorname{grad} w \quad = \quad f$$

scalar advection-diffusion.

Advection-diffusion of differential forms: in coordinates

$$\mathsf{L}_\beta w + \Delta_{HL} w = f.$$

$$
\begin{array}{ccccccccc}
0 & & C^\infty(\Omega) & \underset{-\operatorname{div}}{\overset{\operatorname{grad}}{\rightleftarrows}} & C^\infty(\Omega;\mathbb{R}^3) & \underset{\operatorname{curl}}{\overset{\operatorname{curl}}{\rightleftarrows}} & C^\infty(\Omega;\mathbb{R}^3) & & C^\infty(\Omega) & & 0 \\
0 & & C^\infty(\Omega) & \xleftarrow{\cdot\beta} & C^\infty(\Omega;\mathbb{R}^3) & \xleftarrow{\times\beta} & C^\infty(\Omega;\mathbb{R}^3) & & C^\infty(\Omega) & & 0
\end{array}
$$

$$(d^{k-1} i_\beta^k \quad + \quad i_\beta^{k+1} d^k) w \quad + \quad (d^{k-1} d_{k-1}^* \quad + \quad d_k^* d^k) w \quad = \quad f$$

$$\operatorname{grad}(\boldsymbol{\beta} \cdot \boldsymbol{A}) \quad - \quad \boldsymbol{\beta} \times (\operatorname{curl} \boldsymbol{A}) \quad + \quad (-\operatorname{grad}\operatorname{div} \quad + \quad \operatorname{curl}\operatorname{curl})\boldsymbol{A} \quad = \quad \boldsymbol{f}$$

advection-diffusion of magnetic potential.

Advection-diffusion of differential forms: in coordinates

$$\mathrm{L}_\beta w + \Delta_{HL} w = f.$$

$$\begin{array}{ccccccc} 0 & C^\infty(\Omega) & C^\infty(\Omega;\mathbb{R}^3) & \underset{\text{curl}}{\overset{\text{curl}}{\rightleftarrows}} & C^\infty(\Omega;\mathbb{R}^3) & \underset{-\,\text{grad}}{\overset{\text{div}}{\rightleftarrows}} & C^\infty(\Omega) \quad 0 \\ 0 & C^\infty(\Omega) & C^\infty(\Omega;\mathbb{R}^3) & \xleftarrow{\times\beta} & C^\infty(\Omega;\mathbb{R}^3) & \xleftarrow{\otimes\beta} & C^\infty(\Omega) \quad 0 \end{array}$$

$$(d^{k-1} i_{\boldsymbol{B}}^{k} \quad + \quad i_{\boldsymbol{B}}^{k+1} d^k) w \quad + \quad (d^{k-1} d_{k-1}^{*} \quad + \quad d_k^* d^k) w \quad = \quad f$$

$$-\operatorname{curl}(\beta \times \boldsymbol{B}) \quad + \quad (\operatorname{div} \boldsymbol{B})\beta \quad + \quad (\operatorname{curl}\operatorname{curl} \quad - \quad \operatorname{grad}\operatorname{div})\boldsymbol{B} \quad = \quad \boldsymbol{f}$$

If imposing $\operatorname{div} \boldsymbol{B} = 0$:

$$-\operatorname{curl}(\beta \times \boldsymbol{B}) + \operatorname{curl}\operatorname{curl} \boldsymbol{B} = \boldsymbol{f}.$$

magnetic advection-diffusion.

Advection-diffusion of differential forms: in coordinates

$$\mathsf{L}_\beta w + \Delta_{HL} w = f.$$

$$\begin{array}{ccccccc}
0 & C^\infty(\Omega) & C^\infty(\Omega;\mathbb{R}^3) & C^\infty(\Omega;\mathbb{R}^3) & \underset{-\operatorname{grad}}{\overset{\operatorname{div}}{\rightleftarrows}} & \color{olive}{C^\infty(\Omega)} & \rightleftarrows 0 \\
0 & C^\infty(\Omega) & C^\infty(\Omega;\mathbb{R}^3) & C^\infty(\Omega;\mathbb{R}^3) & \xleftarrow{\otimes\beta} & C^\infty(\Omega) & \longleftarrow 0
\end{array}$$

$$(d^{k-1} i_{\boldsymbol{B}}^k \quad + \quad i_{\boldsymbol{B}}^{k+1} d^k) w \quad + \quad (d^{k-1} d_{k-1}^* \quad + \quad d_k^* d^k) w \quad = \quad f$$

$$\operatorname{div}(u\beta) \qquad - \qquad \operatorname{div}\operatorname{grad} u \quad = \quad f$$

Fokker-Planck type equation (transport of density)

Convergence of Advection-Diffusion Eigenvalue Problems

Find $\boldsymbol{B} \in V^k$, $\lambda \in \mathbb{C}$:

$$(\delta \boldsymbol{B}, \delta \boldsymbol{C}) + (i_{\boldsymbol{u}} \boldsymbol{B}, \delta \boldsymbol{C}) = \lambda(\boldsymbol{B}, \boldsymbol{C}), \quad \forall \boldsymbol{C} \in V^k$$

Find $\boldsymbol{B} \in H(\text{curl})$, $\lambda \in \mathbb{C}$:

$$R_m^{-1}(\nabla \times \boldsymbol{B}, \nabla \times \boldsymbol{C}) - (\boldsymbol{u} \times \boldsymbol{B}, \nabla \times \boldsymbol{C}) = \lambda(\boldsymbol{B}, \boldsymbol{C}), \quad \forall \boldsymbol{C} \in H(\text{curl})$$

Bramble-Osborn Theory: Under assumptions

- *Solution operator* $T : X \to X$ is compact

  $T : \boldsymbol{f} \mapsto \boldsymbol{B}$ solves $R_m^{-1}(\nabla \times \boldsymbol{B}, \nabla \times \boldsymbol{C}) - (\boldsymbol{u} \times \boldsymbol{B}, \nabla \times \boldsymbol{C}) = (\boldsymbol{f}, \boldsymbol{C})$.

- $T_h : X_h \to X_h$ is compact and finite rank

$$\|T - T_h\| \to 0 \implies \text{convergence}$$

James Bramble

John Osborn

Application to MHD: Boils down to regularity of T: $V_0 := T(L^2) \hookrightarrow\hookrightarrow H(\text{curl})$

Theorem [KH, Liang, Zerbinati]: For given **smooth** $\boldsymbol{u}$, $V_0 \hookrightarrow\hookrightarrow H(\text{curl}) \implies$ eigenvalue convergence

Rayleigh quotient (min-max) fails due to non-self-adjoint advection, losing information (e.g., convergence of individual eigenvalues with multiplicity)

Witten transform: when wind is potential (gradient)

$$\begin{tikzcd}
\cdots \arrow[r] & \Lambda^{k-1} \arrow[r, "d"] \arrow[d, "e^{\theta(x)}"] & \Lambda^{k} \arrow[r, "d"] \arrow[d, "e^{\theta(x)}"] & \Lambda^{k+1} \arrow[r, "d"] \arrow[d, "e^{\theta(x)}"] & \cdots \\
\cdots \arrow[r] & \Lambda^{k-1} \arrow[r, "d_\theta"] & \Lambda^{k} \arrow[r, "d_\theta"] & \Lambda^{k+1} \arrow[r, "d_\theta"] & \cdots
\end{tikzcd}$$

Diagram commutes:

$$e^{\theta(\boldsymbol{x})} d(e^{-\theta(\boldsymbol{x})} w) = -\nabla\theta \wedge w + dw.$$

gauge transform. Compare to covariant derivatives $\nabla w = \partial w + \Gamma \cdot w$.

$$\begin{tikzcd}
\cdots & \Lambda^{k-1} \arrow[l] \arrow[d, "e^{\theta(x)}"] & \Lambda^{k} \arrow[l, "\delta"'] \arrow[d, "e^{\theta(x)}"] & \Lambda^{k+1} \arrow[l, "\delta"'] \arrow[d, "e^{\theta(x)}"] & \cdots \arrow[l, "\delta"'] \\
\cdots & \Lambda^{k-1} \arrow[l] & \Lambda^{k} \arrow[l, "\delta_\theta"'] & \Lambda^{k+1} \arrow[l, "\delta_\theta"'] & \cdots \arrow[l, "\delta_\theta"']
\end{tikzcd}$$

$$\delta_\theta u := e^{-\theta(\boldsymbol{x})} \delta e^{\theta(\boldsymbol{x})} u = \iota_{\pm(d\theta)^\sharp} u + \delta u.$$

$$\boxed{d\delta_{\pm\theta} + \delta_{\pm\theta} d = \Delta_{HL} + \mathrm{L}_{\nabla\theta}}$$

Hodge Laplacian on transformed coordinates $=$ advection-diffusion

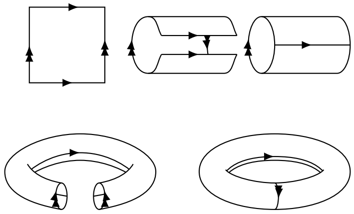
Numerical applications: stablizing numerical oscillation Brezzi,Marini,Pietra 1989 : exponential fitting, scalar problem; Wu,Xu 2018 : forms in 3D; Christiansen,Halvorsen,Sørensen 2014 : Petrov Galerkin

Edward Witten

Supersymmetry and Morse theory, Witten (1982) J. Diff. Geo.

Witten deformation
Witten complex
Witten Laplacian

Consequence 1: for potential (gradient) winds, eigenvalues are real.



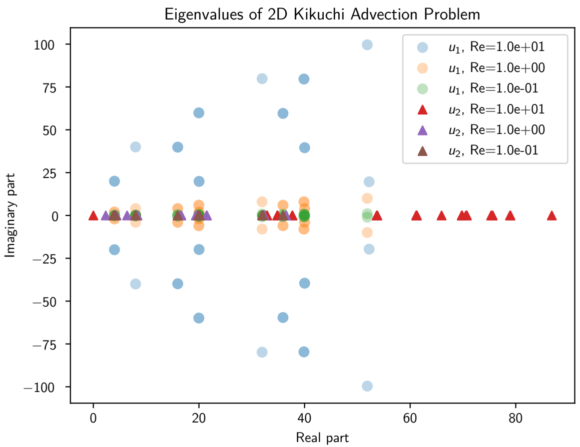


Figure. Eigenvalues for the toroidal surface with wind $\mathbf{u}_1$ (non-gradient) and $\mathbf{u}_2$ (gradient).

$$\mathbf{u}_1 = (1, 1), \quad \mathbf{u}_2 = (2\cos(2x)\sin(2y), 2\sin(2x)\cos(2y))$$

Consequence 2: improved estimates (essentially self-adjoint)

Generalizing Hodge theory

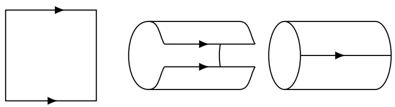
Theorem 2 (V. I. Arnold)

The number of linearly independent stationary k-forms is **not less than** *the k-th betti number of the manifold* $\mathcal{M}$.

Theorem 3 (V. I. Arnold)

If the diffusion coefficient R_m^{-1} *is sufficiently large, then the number of linearly independent stationary k-forms is* **equal to** *the k-th betti number of the manifold* $\mathcal{M}$.

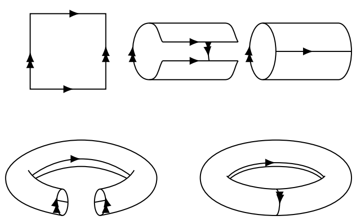
Test 1



$b_0 = 1, b_1 = 1, b_2 = 0.$ $\mathbf{u}_1 = (1, 1)$

R_m	100	10	1	0.1
$\dim(\lambda_0)$	1	1	1	1

Test 2



$b_0 = 1, b_1 = 2, b_2 = 1.$
$\mathbf{u}_2 = (2\cos(2x)\sin(2y), 2\sin(2x)\cos(2y))$

R_m	100	10	1	0.1
$\dim(\lambda_0)$	2	2	2	2

Back to the very first assumption...

> *Numerically,* **the kinematic fast dynamo problem is the first eigenvalue problem** *for matrices of the order of many million, even for reasonable Reynolds numbers (of the order of hundreds). The physically meaningful magnetic Reynolds numbers R_m are of order of magnitude 10^8. The corresponding matrices are (and will remain) beyond the reach of any computer.*

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— *Topological Methods in Hydrodynamics*, V.I.Arnold, B.A.Khesin 2021.

Eigenvalue analysis is often **misleading** for telling (in)stability.

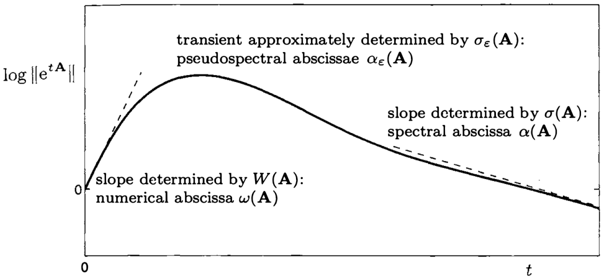
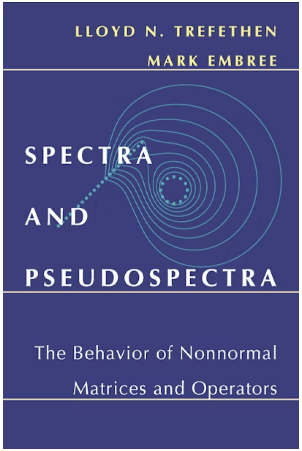


Figure 14.1: Initial, transient, and asymptotic behavior of $\|e^{t\mathbf{A}}\|$ for a nonnormal matrix or operator $\mathbf{A}$.



In transient region (better described by pseudo-spectra), non-linear effects become dominating, Asymptotics described by eigenvalues never reached .

Structure-preserving FE also computes pseudo-spectra

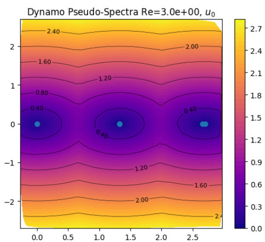
$$\sigma_\epsilon(\boldsymbol{A}) = \{z \in \mathbb{C} : \|(z - \boldsymbol{A})^{-1}\| > \epsilon^{-1}\}$$

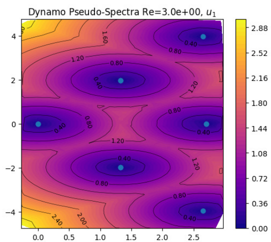
$$\sigma_\epsilon(\boldsymbol{A}) = \{z \in \mathbb{C} : s_{\min}(z - \boldsymbol{A}) < \epsilon\}$$

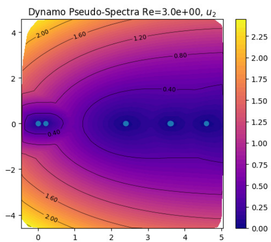
$s_{\min}$: minimal singular value

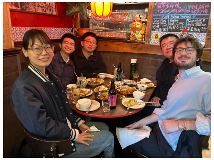
Theorem 4 (Zerbinati)

Finite element for pseudo-spectra converges.









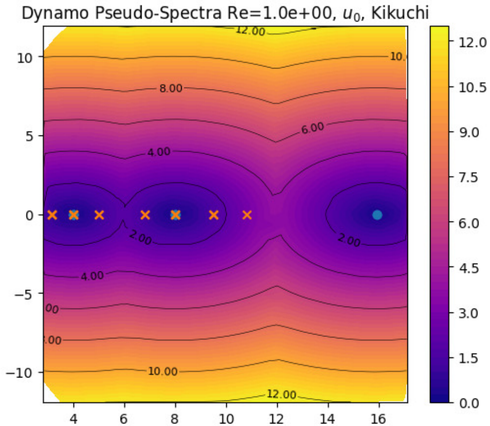
Umberto Zerbinati,
Mini-course
Edinburgh 2025

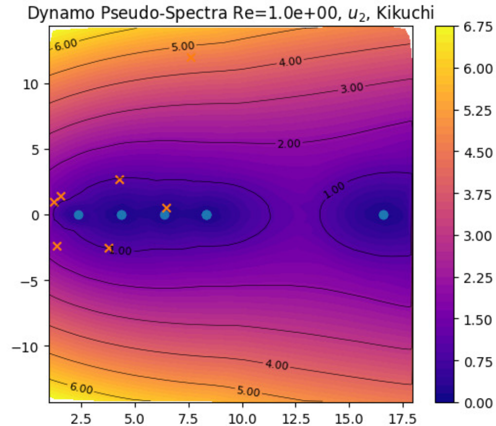
$$\mathbf{u}_0 = 0 \quad \text{Hodge Laplacian}$$

$$\mathbf{u}_1 = (1, 1) \quad \text{non potential}$$

$$\mathbf{u}_2 = (2\cos(2x)\sin(2y), 2\sin(2x)\cos(2y)) \quad \text{potential}$$

Lagrange elements lead to spurious modes again





$\mathbf{u}_0 = 0$ Hodge Laplacian

$\mathbf{u}_1 = (1, 1)$ non potential

×: from nodal elements ○: Nédélec (FEEC)

Back to Arnold&Korkina 1983 computation

Arnold, V. I., & Korkina, E. I. (1983). *The growth of a magnetic field in the three-dimensional steady flow of an incompressible fluid*. Moskovskii Universitet Vestnik Seriia Matematika Mekhanika, 43-46.

Fourier basis $e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = e^{i(k_1x_1+\cdots+k_nx_n)}$

$$0 \longrightarrow e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \xrightarrow{\text{grad}} \begin{pmatrix} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \\ e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \\ e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \end{pmatrix} \xrightarrow{\text{curl}} \begin{pmatrix} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \\ e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \\ e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \end{pmatrix} \xrightarrow{\text{div}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \longrightarrow 0.$$

“good” complex. Ongoing work with A. Bressan, Y. Zhu.

Compared to **finite element** exterior calculus, less attention paid to **spectral** basis and spectral methods.

Summary

An exciting start...

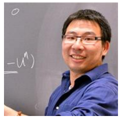
FEEC for topological hydrodynamics

- Long-term evolution and rough solutions.
- Solvers.
- Nonlinear eigenvalue problems, pseudo-spectra...
- Turbulence.
- Going beyond compactness framework in computing eigenvalues.
- ...

Collaborators: Algorithms & Analysis

Jinchao Xu
Penn State, KAUST

Yicong Ma
Barclays



Xiaozhe Hu
Tufts

Young-Ju Lee
Texas

Patrick Farrell
Oxford

Fabian Laakman
ASML

Lourenco Beirão da Veiga
Milano

Lorenzo Mascotto
Milano

- *Stable finite element methods preserving $\nabla \cdot \boldsymbol{B} = 0$ exactly for MHD models*, K. Hu, Y. Ma, J. Xu; Numerische Mathematik, 135(2), 371-396 (2017). divergence-free preservation
- *Robust preconditioners for incompressible MHD models*, Y. Ma, K. Hu, X. Hu, J. Xu; Journal of Computational Physics, 316, 721-746 (2016). preconditioning
- *Helicity-conservative finite element discretization for incompressible MHD systems*, K. Hu, Y.-J. Lee, J. Xu; Journal of Computational Physics (2021). helicity preservation
- *Structure-preserving and helicity-conserving finite element approximations and preconditioning for the Hall MHD equations*, F. Laakmann, K. Hu, P. E. Farrell; Journal of Computational Physics (2023). Hall MHD
- *Finite element exterior calculus for multiphysics problems*, K. Hu; Peking University (2017) PhD thesis

Collaborators: Relaxation


Mingdong He
Oxford


Patrick Farrell
Oxford


Boris Andrews
Oxford

- *Topology-preserving discretization for the magneto-frictional equations arising in the Parker conjecture*, M. He, P. E. Farrell, K. Hu, B. D. Andrews; SISC (2025)

Collaborators: Dynamo


Daniele Boffi
KAUST


Yizhou Liang
Oxford


Stefano Zampini
KAUST


Umberto Zerbinati
Oxford


Jindong Wang
KAUST, incoming Newton Fellow

- *FEEC for dynamo*, D. Boffi, K. Hu, Y. Liang, S. Zampini, U. Zerbinati; in preparation
- *Pseudospectra of advection-diffusion of differential forms*, D. Boffi, K. Hu, U. Zerbinati; in preparation

Aknowledgement

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