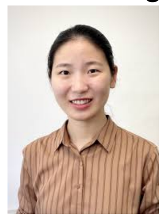
Finite element form-valued forms
– Towards an extended periodic table –

Kaibo Hu

joint work with Ting Lin (Peking), Qian Zhang (MichiganTech)

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Distributional Hessian and Divdiv Complexes on Triangulation and Cohomology*

Kaibo Hu†, Ting Lin‡, and Qian Zhang§

Abstract. We construct discrete versions of some Bernstein–Gelfand–Gelfand (BGG) complexes, i.e., the Hessian and the divdiv complexes, on triangulations in two dimensions and three dimensions. The sequences consist of finite elements with local polynomial shape functions and various types of Dirac measures on subsimplices (generalizations of currents). The construction generalizes Whitney forms (canonical conforming finite elements) for the de Rham complex and Regge calculus/finite elements for the elasticity (Riemannian deformation) complex from discrete topological and Discrete Exterior Calculus perspectives. We show that the cohomology of the resulting complexes is isomorphic to the continuous versions, and thus isomorphic to the de Rham cohomology with coefficients.

Key words. Bernstein–Gelfand–Gelfand sequences, cohomology, finite element exterior calculus, discrete exterior calculus, currents

- KH, Ting Lin. *Finite element form-valued forms (I): Construction*. arXiv: 2503.03243 (2025) *n*D.

FINITE ELEMENT FORM-VALUED FORMS (I): CONSTRUCTION

KAIBO HU AND TING LIN

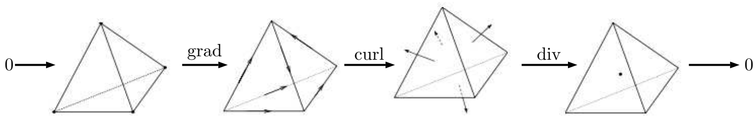
ABSTRACT. We provide a finite element discretization of ℓ-form-valued k-forms on triangulation in $\mathbb{R}^n$ for general k, ℓ and n and any polynomial degree. The construction generalizes finite element Whitney forms for the de Rham complex and their higher-order and distributional versions, the Regge finite elements and the Christiansen–Regge elasticity complex, the TDNNS element for symmetric stress tensors, the MCS element for traceless matrix fields, the Hellan–Herrmann–Johnson (HHJ) elements for biharmonic equations, and discrete divdiv and Hessian complexes in [Hu, Lin, and Zhang, 2025]. The construction discretizes the Bernstein–Gelfand–Gelfand (BGG) diagrams. Applications of the construction include discretization of strain and stress tensors in continuum mechanics and metric and curvature tensors in differential geometry in any dimension.

MOTIVATION

stress, strain tensors, dislocation density, disclination density in continuum mechanics,
metric, curvature (scalar, Ricci, Weyl, Riemann, Cotton...), torsion in differential geometry etc.

Are there discrete analogues of such tensors with symmetries and differential structures?

A special case: differential forms (fully skew-symmetric tensors), exterior derivatives



Whitney forms and higher order versions are well accepted as the canonical discretization for differential forms (skew-symmetric tensors). k-forms discretized on k-cells, unisolvent with $\mathcal{P}^-\Lambda^k$, conformity

Deriving complexes: complexes from complexes

Inspired by the Bernstein-Gelfand-Gelfand (BGG) construction. (B-G-G 1975, Čap,Slovák,Souček 2001, Eastwood 2000, Arnold,Falk,Winther 2006, Arnold,KH 2021, Čap,KH 2023)

$$
\begin{tikzcd}
0 \arrow[r] & \mathrm{Alt}^{0,J-1} \arrow[r, "d"] & \mathrm{Alt}^{1,J-1} \arrow[r, "d"] & \cdots \arrow[r, "d"] & \mathrm{Alt}^{n,J-1} \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Alt}^{0,J} \arrow[r, "d"] \arrow[ur, "S^{0,J}"] & \mathrm{Alt}^{1,J} \arrow[r, "d"] \arrow[ur, "S^{1,J}"] & \cdots \arrow[r, "d"] \arrow[ur, "S^{n-1,J}"] & \mathrm{Alt}^{n,J} \arrow[r] & 0
\end{tikzcd}
$$

where $\mathrm{Alt}^{i,J} := \mathrm{Alt}^{i} \otimes \mathrm{Alt}^{J}$ J-form-valued i-forms (i-forms taking value in J-forms; (i, J)-form)

$$S^{i,J}\mu(v_0, \cdots, v_i)(w_1, \cdots, w_{J-1}) := \sum_{l=0}^{i}(-1)^{l}\mu(v_0, \cdots, \widehat{v_l}, \cdots, v_i)(v^{l}, w_1, \cdots, w_{J-1}),$$

$$S_{\dagger}^{i,J}\mu(v_1, \cdots, v_{i-1})(w_0, \cdots, w_J) := \sum_{l=0}^{i}(-1)^{l}\mu(w_l, v_1, \cdots, v_{i-1})(w_1, \cdots, \widehat{w_l}, \cdots, w_J),$$

$$\forall v_0, \cdots, v_i, w_1, \cdots, w_J \in \mathbb{R}^n.$$

Output:

$$
\begin{tikzcd}
0 \arrow[r] & \ker(S_{\dagger}^{-1,J}) \arrow[r] & \cdots \arrow[r] & \ker(S_{\dagger}^{J-2,J}) \arrow[r, "d"] & {} \arrow[dll, "S^{-1}"] & & & \\
& & {} \arrow[r, "d"] & \ker(S^{J,J}) \arrow[r] & \cdots \arrow[r] & \ker(S^{n,J}) \arrow[r] & 0.
\end{tikzcd}
$$

$S_{\dagger}$: adjoint of S

$$
\begin{tikzcd}
 & \cdots \arrow[r] & \cdots \arrow[r] & \cdots \arrow[r] & \cdots & \\
\cdots \arrow[r] & \mathrm{Alt}^{k-1,\ell-1} \arrow[r, "d^{k-1}"] \arrow[ur] & \mathrm{Alt}^{k,\ell-1} \arrow[r, "d^{k}"] \arrow[ur] & \mathrm{Alt}^{k+1,\ell-1} \arrow[r, "d^{k+1}"] \arrow[ur] & \mathrm{Alt}^{k+2,\ell-1} \arrow[r] & \cdots \\
\cdots \arrow[r] & \mathrm{Alt}^{k-1,\ell} \arrow[r, "d^{k-1}"] \arrow[ur, "S^{k-1,\ell}"'] & \mathrm{Alt}^{k,\ell} \arrow[r, "d^{k}"] \arrow[ur, "S^{k,\ell}"'] & \mathrm{Alt}^{k+1,\ell} \arrow[r, "d^{k+1}"] \arrow[ur, "S^{k+1,\ell}"'] & \mathrm{Alt}^{k+2,\ell} \arrow[r] & \cdots \\
\cdots \arrow[r] & \mathrm{Alt}^{k-1,\ell+1} \arrow[r, "d^{k-1}"] \arrow[ur, "S^{k-1,\ell+1}"'] & \mathrm{Alt}^{k,\ell+1} \arrow[r, "d^{k}"] \arrow[ur, "S^{k,\ell+1}"'] & \mathrm{Alt}^{k+1,\ell+1} \arrow[r, "d^{2}"] \arrow[ur, "S^{k+1,\ell+1}"'] & \mathrm{Alt}^{k+2,\ell+1} \arrow[r] & \cdots \\
\cdots \arrow[r] & \mathrm{Alt}^{k-1,\ell+2} \arrow[r, "d^{k-1}"] \arrow[ur, "S^{k-1,\ell+2}"'] & \mathrm{Alt}^{k,\ell+2} \arrow[r, "d^{k}"] \arrow[ur, "S^{k,\ell+2}"'] & \mathrm{Alt}^{k+1,\ell+2} \arrow[r, "d^{k+1}"] \arrow[ur, "S^{k+1,\ell+2}"'] & \mathrm{Alt}^{k+2,\ell+2} \arrow[r] & \cdots \\
 & \cdots \arrow[r] \arrow[ur] & \cdots \arrow[r] \arrow[ur] & \cdots \arrow[r] \arrow[ur] & \cdots &
\end{tikzcd}
$$

Tensor symmetries encoded in diagrams: e.g.,

- $\ker(S_{\dagger}^{0,2}) \subset \mathrm{Alt}^{1,1}$: symmetric 2-tensors (metrics)
- $\ker(S^{2,2}) \subset \mathrm{Alt}^{2,2}$: symmetry of Riemannian tensor (algebraic Bianchi identity)
  $R_{abcd} = -R_{bacd} = -R_{abdc}, \quad R_{abcd} = R_{cdab}, \quad R_{abcd} + R_{bcad} + R_{cabd} = 0.$

$$
\dim(\ker(S^{2,2})) = \dim \mathrm{Alt}^{2,2} - \dim \mathrm{Alt}^{3,1} = \begin{cases} 1 & \text{in 2D} \quad \text{Gauss curvature,} \\ 6 & \text{in 3D} \quad \text{Ricci/Einstein,} \\ 20 & \text{in 4D} \quad \text{Riemann} \\ \cdots & \end{cases}
$$

BGG construction: connecting any row k and row ℓ of the above diagram, zig-zag at $\mathrm{Alt}^{k.\ell} \cong \mathrm{Alt}^{\ell,k}$

3D EXAMPLES

$\mathbb{R}$: scalar $\mathbb{V}$: vector $\mathbb{M}$: matrix $\mathbb{S}$: symmetric matrix $\mathbb{T}$: trace-free matrix

$$
\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r, "\mathrm{grad}"] & \mathbb{V} \arrow[r, "\mathrm{curl}"] & \mathbb{V} \arrow[r, "\mathrm{div}"] & \mathbb{R} \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "\mathrm{id}"] & \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "2\,\mathrm{vskw}"] & \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{tr}"] & \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\mathrm{mskw}"'] & \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "S"'] & \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "2\,\mathrm{vskw}"'] & \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{R} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\iota"] & \mathbb{V} \arrow[r, "\mathrm{curl}"] \arrow[ur, "\mathrm{mskw}"] & \mathbb{V} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{id}"] & \mathbb{R} \arrow[r] & 0.
\end{tikzcd}
$$

Hessian complex:

$$0 \longrightarrow C^{\infty} \xrightarrow{\mathrm{hess}} C^{\infty}(\mathbb{S}) \xrightarrow{\mathrm{curl}} C^{\infty}(\mathbb{T}) \xrightarrow{\mathrm{div}} C^{\infty}(\mathbb{V}) \longrightarrow 0.$$

biharmonic equations, plate theory, Einstein-Bianchi system of general relativity

3D EXAMPLES

$\mathbb{R}$: scalar $\quad \mathbb{V}$: vector $\quad \mathbb{M}$: matrix $\quad \mathbb{S}$: symmetric matrix $\quad \mathbb{T}$: trace-free matrix

$$
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0 \arrow[r] & \mathbb{R} \arrow[r, "\mathrm{grad}"] & \mathbb{V} \arrow[r, "\mathrm{curl}"] & \mathbb{V} \arrow[r, "\mathrm{div}"] & \mathbb{R} \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "\mathrm{id}"] & \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "2\,\mathrm{vskw}"] & \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{tr}"] & \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\,\mathrm{mskw}"] & \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "S"] & \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "2\,\mathrm{vskw}"] & \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{R} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\iota"] & \mathbb{V} \arrow[r, "\mathrm{curl}"] \arrow[ur, "\mathrm{mskw}"] & \mathbb{V} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{id}"] & \mathbb{R} \arrow[r] & 0.
\end{tikzcd}
$$

elasticity complex:

$$0 \longrightarrow C^\infty(\mathbb{V}) \xrightarrow{\mathrm{def}} C^\infty(\mathbb{S}) \xrightarrow{\mathrm{inc}} C^\infty(\mathbb{S}) \xrightarrow{\mathrm{div}} C^\infty(\mathbb{V}) \longrightarrow 0.$$

elasticity, defects, metric, curvature

3D EXAMPLES

$\mathbb{R}$: scalar $\mathbb{V}$: vector $\mathbb{M}$: matrix $\mathbb{S}$: symmetric matrix $\mathbb{T}$: trace-free matrix

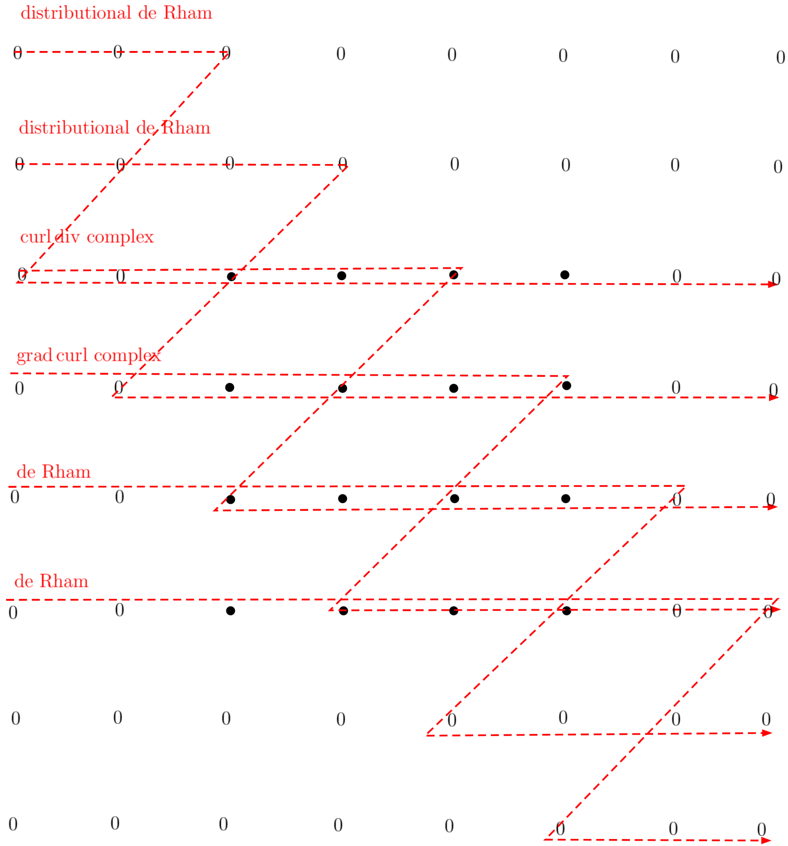
$$
\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r, "\mathrm{grad}"] & \mathbb{V} \arrow[r, "\mathrm{curl}"] & \mathbb{V} \arrow[r, "\mathrm{div}"] & \mathbb{R} \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "\mathrm{id}"] & \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "2\,\mathrm{vskw}"] & \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{tr}"] & \mathbb{V} \arrow[r] & 0 \\
{\color{blue}0} \arrow[r] & {\color{blue}\mathbb{V}} \arrow[r, "{\color{blue}\mathrm{grad}}"] \arrow[ur, "-\,\mathrm{mskw}"] & {\color{blue}\mathbb{M}} \arrow[r, "{\color{blue}\mathrm{curl}}"] \arrow[ur, "S"] & {\color{blue}\mathbb{M}} \arrow[r, "{\color{blue}\mathrm{div}}"] \arrow[ur, "2\,\mathrm{vskw}"] & {\color{blue}\mathbb{V}} \arrow[r] & {\color{blue}0} \\
{\color{blue}0} \arrow[r] & {\color{blue}\mathbb{R}} \arrow[r, "{\color{blue}\mathrm{grad}}"] \arrow[ur, "{\color{blue}-\iota}"] & {\color{blue}\mathbb{V}} \arrow[r, "{\color{blue}\mathrm{curl}}"] \arrow[ur, "{\color{blue}\mathrm{mskw}}"] & {\color{blue}\mathbb{V}} \arrow[r, "{\color{blue}\mathrm{div}}"] \arrow[ur, "{\color{blue}\mathrm{id}}"] & {\color{blue}\mathbb{R}} \arrow[r] & {\color{blue}0.}
\end{tikzcd}
$$

divdiv complex:

$$0 \longrightarrow C^\infty(\mathbb{V}) \xrightarrow{\text{dev grad}} C^\infty(\mathbb{T}) \xrightarrow{\text{sym curl}} C^\infty(\mathbb{S}) \xrightarrow{\text{div div}} C^\infty \longrightarrow 0.$$

plate theory, elasticity

Iterated constructions: connecting any two rows



Discretization of complexes: finite elements and splines

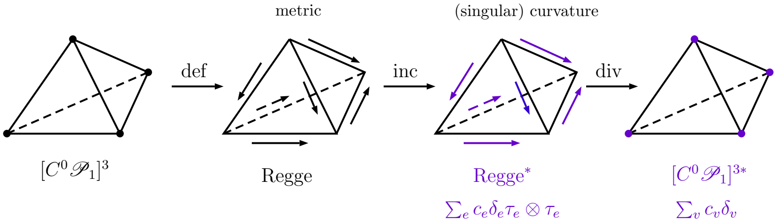
- 2D stress: Arnold-Winther 2002, J.Hu-S.Zhang 2014, Christiansen-KH 2018,
- 2D strain: Chen-J.Hu-Huang 2014 (Regge/HHJ), Christiansen-KH 2018 (conforming), Chen-Huang 2020, DiPietro-Droniou 2021 (polygonal meshes), KH 2023
- 3D elasticity: various results on last part of complex, Hauret-Kuhl-Ortiz 2007 (discrete geometry/mechanics), Arnold-Awanou-Winther 2008, Christiansen 2011 (Regge), Christiansen-Gopalakrishnan-Guzmán-KH 2020, Chen-Huang 2021, J.Hu-Liang-Lin 2023, Gong-Gopalakrishnan-Guzmán-Neilan 2023
- 3D Hessian: Chen-Huang 2020, J.Hu-Liang 2021, Arf-Simeon 2021 (splines)
- 3D divdiv: Chen-Huang 2021, J.Hu-Liang-Ma 2021, Sander 2021 ($H(\mathrm{sym\,curl})$, $H(\mathrm{dev\,sym\,curl})$), J.Hu-Liang-Ma-Zhang 2022, J.Hu-Liang-Lin 2023, DiPietro-Hanot 2023
- nD: Chen-Huang 2021 (last two spaces), 2D arbitrary regularity: Chen-Huang 2022, Bonizzoni-KH-Kanschat-Sap 2023 (tensor product construction, nD, (k, ℓ)-forms)
- conformal complexes: KH-Lin-Shi 2023.

Question: Canonical finite elements for form-valued forms and BGG complexes on triangulation (analogue of the Whitney forms)?

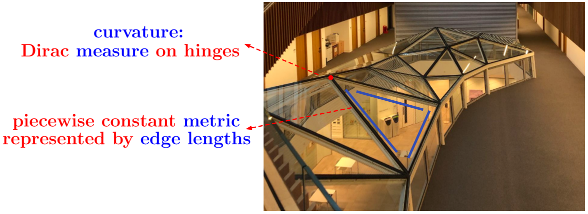
Gathering pieces to solve the puzzle

As the development of FEEC, several individual ingredients are already in the literature. Particularly, it is perhaps with little hesitation to accept Christiansen-Regge complex as the *canonical* discretization for the elasticity complex, due to simple dofs, geometric interpretations, formal self-adjointness, correct cohomology.

Christiansen 2011: Regge calculus = finite elements



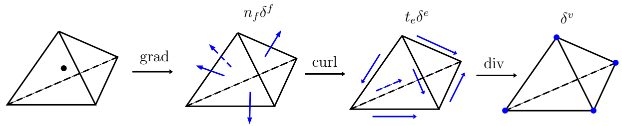
Regge finite element v.s. Regge Calculus (*'General relativity without coordinates'*)



nD: Lizao Li (2018 Minnesota thesis), nonlinear curvature with Regge elements (Berchenko-Kogan,Gawlik 2022, Gopalakrishnan,Neunteufel,Schöberl,Wardetzky 2022, Gawlik,Neunteufel 2023)

Schöberl and collaborators systematically used finite elements + distributions to design numerical schemes.

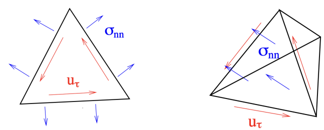
- Equilibrated residual error estimator



distributional finite elements, Braess-Schöberl 2008, Licht 2017

- Tangential Displacement Normal-Normal Stress (TDNNS) method for elasticity

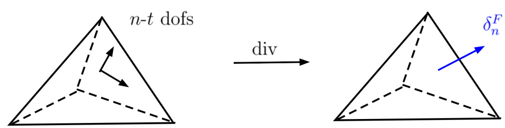
$$\sigma = -\varepsilon(\boldsymbol{u}), \quad \nabla \cdot \sigma = \boldsymbol{f}. \quad \sigma : \text{ symmetric matrix}$$



div σ: tangential distribution Schöberl-Sinwel 2007

- Mass-Conserving mixed Stress (MCS) method for Stokes equations

$$\sigma = -\nabla \boldsymbol{u}, \quad \nabla \cdot \sigma + \nabla p = \boldsymbol{f}. \quad \sigma : \text{ trace-free matrix}$$



div σ: normal distribution Gopalakrishnan-Lederer-Schöberl 2020

WITH SOME NEW INGREDIENTS, PIECES GLUE TOGETHER

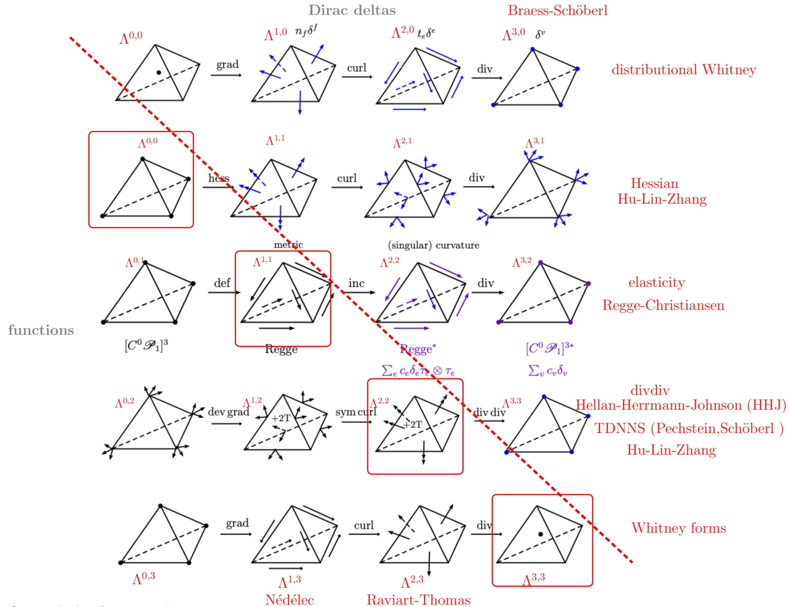
$$\mathrm{Alt}^{-1,-1} \dashrightarrow \quad \swarrow \quad \dashrightarrow \mathrm{Alt}^{0,0} \longrightarrow \mathrm{Alt}^{1,0} \longrightarrow \mathrm{Alt}^{2,0} \longrightarrow \mathrm{Alt}^{3,0}$$

$$\mathrm{Alt}^{0,0} \longrightarrow \quad \swarrow \quad \longrightarrow \mathrm{Alt}^{1,1} \longrightarrow \mathrm{Alt}^{2,1} \longrightarrow \mathrm{Alt}^{3,1}$$

$$\mathrm{Alt}^{0,1} \longrightarrow \mathrm{Alt}^{1,1} \longrightarrow \quad \swarrow \quad \longrightarrow \mathrm{Alt}^{2,2} \longrightarrow \mathrm{Alt}^{3,2}$$

$$\mathrm{Alt}^{0,2} \longrightarrow \mathrm{Alt}^{1,2} \longrightarrow \mathrm{Alt}^{2,2} \longrightarrow \quad \swarrow \quad \longrightarrow \mathrm{Alt}^{3,3}$$

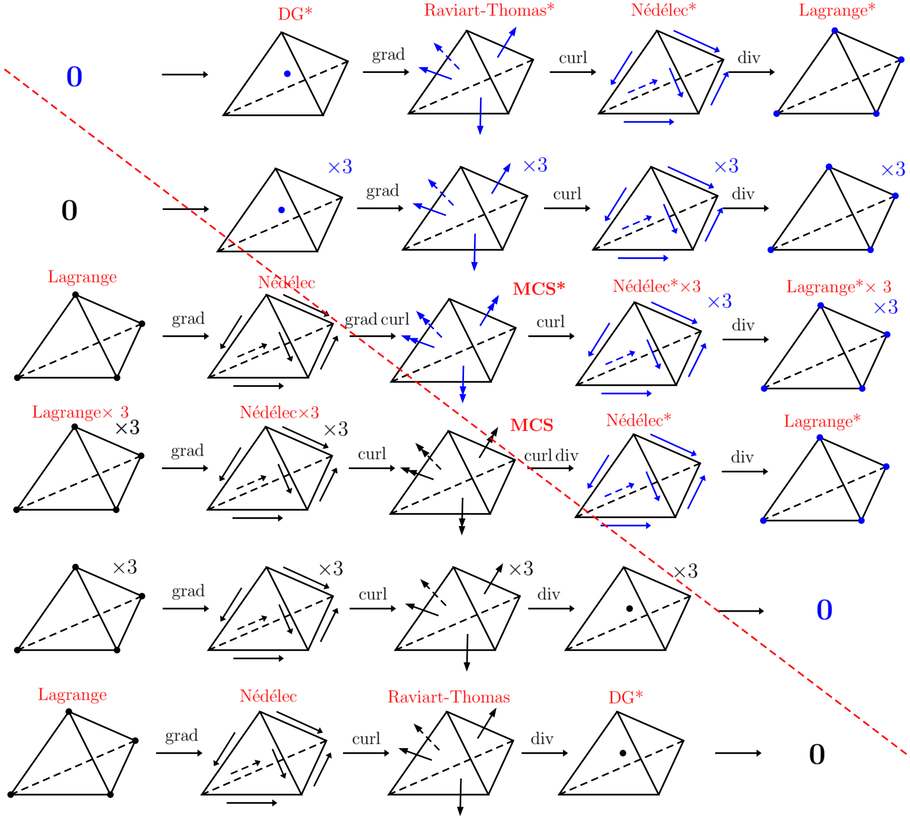
$$\mathrm{Alt}^{0,3} \longrightarrow \mathrm{Alt}^{1,3} \longrightarrow \mathrm{Alt}^{2,3} \longrightarrow \mathrm{Alt}^{3,3} \dashrightarrow \quad \swarrow \quad \dashrightarrow \mathrm{Alt}^{4,4}$$



Patterns for discretization for $\mathrm{Alt}^k \otimes \mathrm{Alt}^\ell$ (ℓ-form-valued k-forms):

- functions before zig-zag, currents after zig-zag
- (roughly speaking) function part: attaching ℓ-forms to k-cells; distributional part (upper triangular): attaching ℓ-forms to $(n-k)$-cells (dual k-cells).
  DEC way of thinking: primal mesh before zig-zag, dual mesh after zig-zag (not needed in this work)
- a lot of duality, correct cohomology (KH. Lin, Zhang 2025; Christiansen, KH, Lin 2024)

WHAT DOES MCS SHOW UP? JUMP TWO ROWS.



Motivation for a unified theory in nD, discretizing the entire diagram

- Problems from differential geometry and relativity require discretizing tensor fields in four and higher dimensions.
- Desirable to discretize the entire diagram, not only the derived complexes : the diagram (twisted complex) encodes more physics (microstructure) and geometry (connection, torsion).
- Cliques (analogues of simplices) of any dimensions exist on graphs .

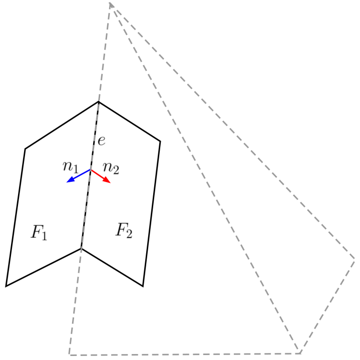
TRACE AND GENERALIZED TRACE: CORRECT CONTINUITY & DOFS FOR $\mathrm{Alt}^{k,\ell}$?

Trace of k-forms on j-cells: projection to the cell

$$\iota_F^*\omega(v_1,\cdots,v_k)=\omega(\iota_* v_1,\cdots,\iota_* v_k),\quad \omega\in C^\infty\mathrm{Alt}^k(\Omega);\, v_1,\cdots,v_k\in\mathscr{X}(\Omega).$$

trace of k-forms vanishes on ℓ-cells if $k>\ell$

Generalized trace $\jmath^*$: *feeding tangent vectors as much as possible; allowing normal vectors if necessary.*
e.g., define trace of 2-forms on 1-cells. Feed 1 tangent vector and 1 normal ($n-1$ choices) to the 2-form.



ω: 2-form, e: edge
Trace $\iota_e^*\omega=0$
Generalized trace $\jmath_e^*\omega\neq 0$,
project to F_1, F_2.

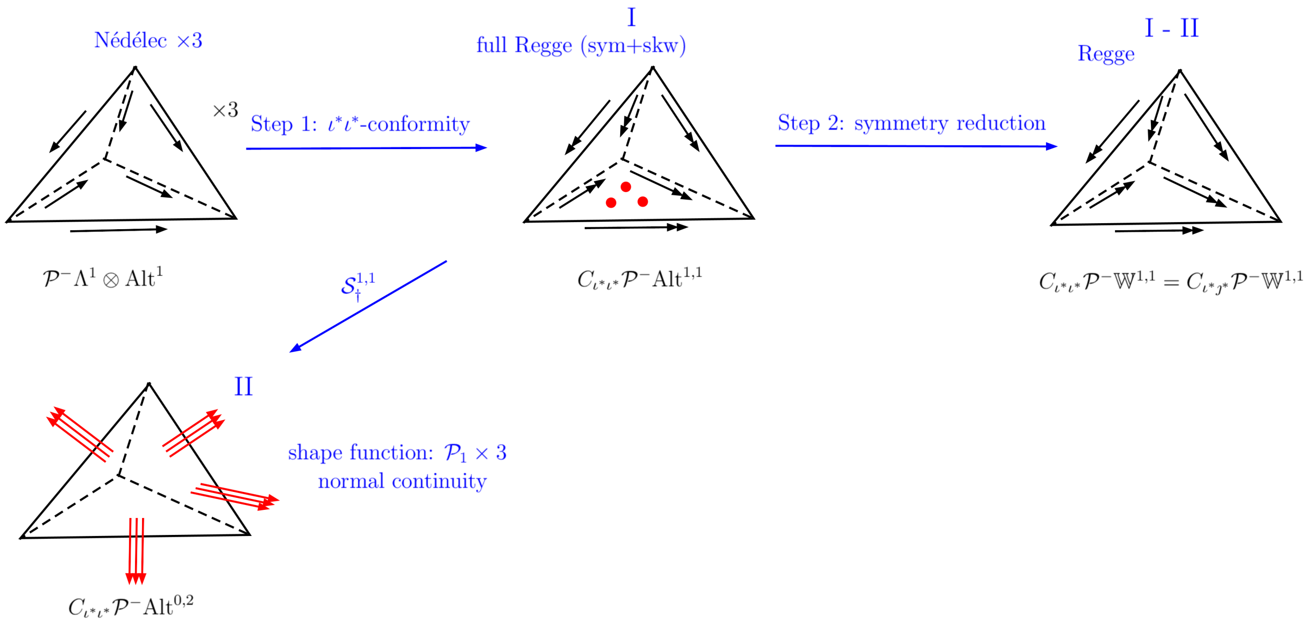
Trace ι^* (above) and generalized trace $\jmath^*$ (below) on k-forms in $\mathbb{R}^3$

k \ dim(e)	0	1	2
0	vertex value	edge value	face value
1	0	edge tangential	face tangential
2	0	0	face normal

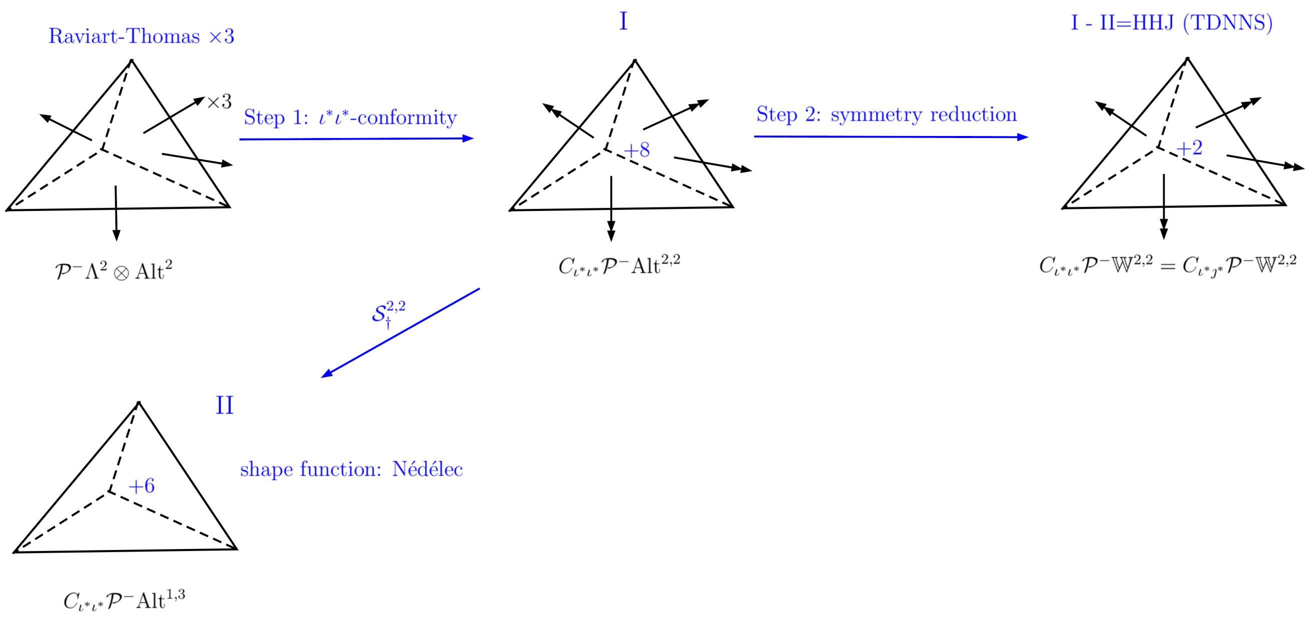
k \ dim(e)	0	1	2
0	vertex value	edge value	face value
1	vertex value	edge tangential	face tangential
2	vertex value	edge normal	face normal

Single-valuedness of the $\iota^*\jmath^*$-trace is the continuity we need for Hess, elasticity, divdiv etc.
For $\mathrm{Alt}^{k,\ell}$: classical trace for the form part k, and enhanced trace for the bundle part ℓ.

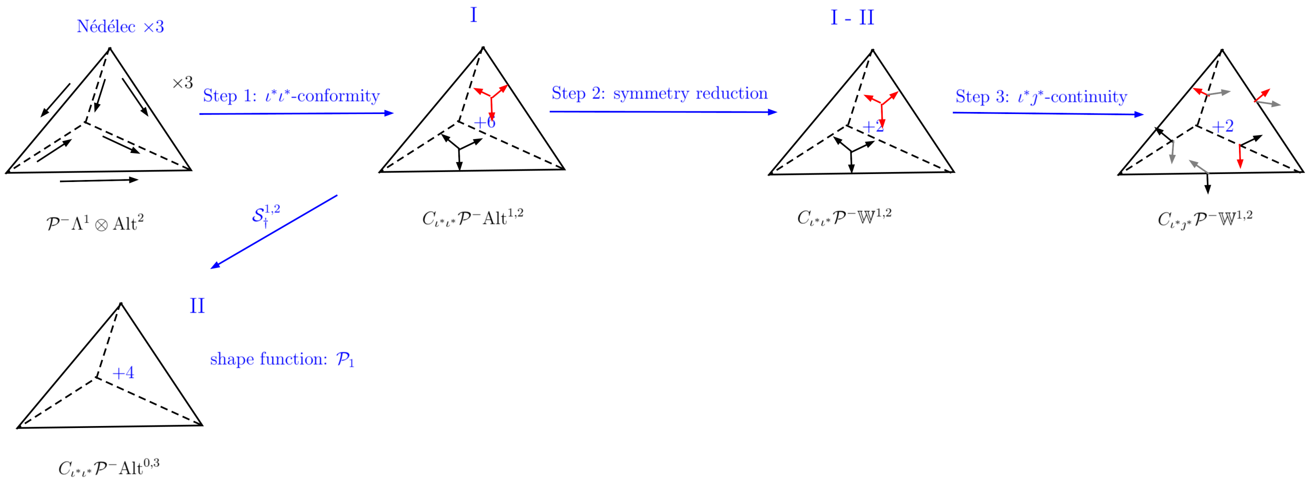
Construction through examples: Regge from Nédélec



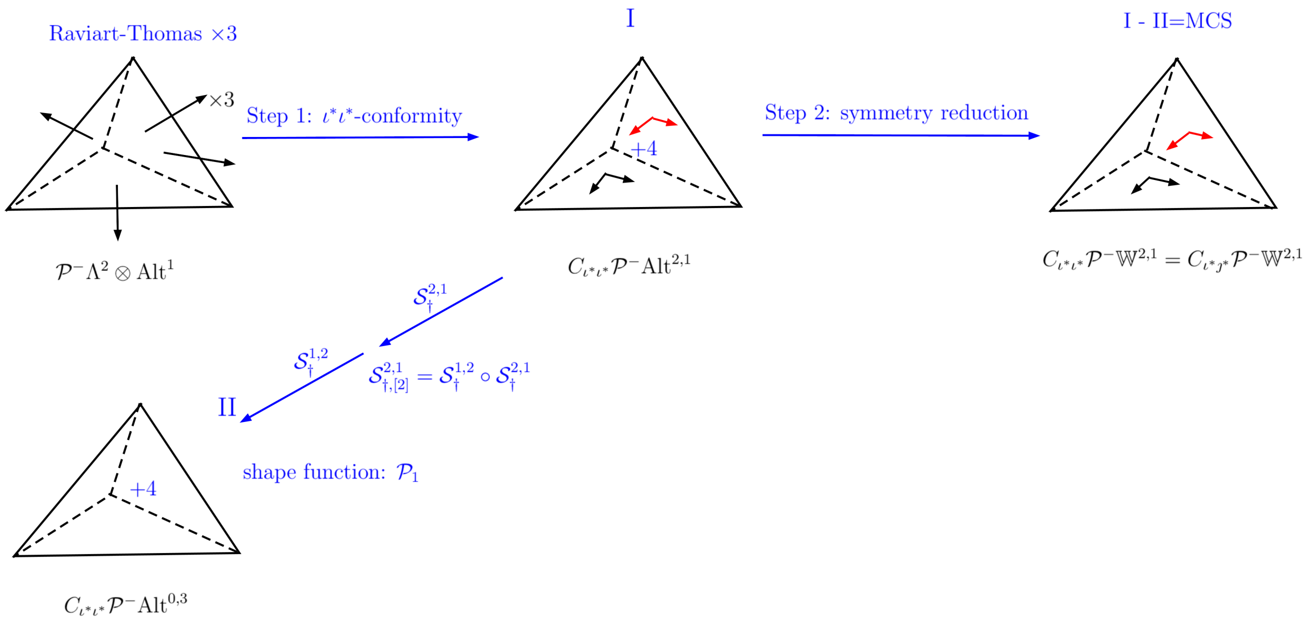
Construction through examples: HHJ/TDNNS from Raviart–Thomas



Construction through examples: HLZ from Nédélec



Construction through examples: MCS from Raviart–Thomas



General construction (lowest order case)

p: jump how many rows ($p = 1$ for Hess, elasticity, divdiv, $p = 2$ for grad curl, curl div etc.)
Function part: For $k \leq \ell + p - 1$, shape functions $\mathcal{P}^-\mathbb{W}^{k,\ell}_{[p]} := \ker(\mathcal{S}^{k,\ell}_{\dagger,[p]} : \mathcal{P}^-\mathrm{Alt}^{k,\ell} \to \mathcal{P}^-\mathrm{Alt}^{k-p,\ell+p})$.

$$\begin{cases} \langle \iota_\sigma^* \jmath_{\sigma,[p]}^* \omega, b\rangle_\sigma, & \forall\ b \in \bigoplus_{s=0}^{p-1} \mathrm{Alt}^{k-s}(\sigma) \otimes \mathrm{Alt}^{\ell-k+s}(\sigma^\perp), \quad \dim\sigma = k \\ \langle \iota_\sigma^* \iota_\sigma^* \omega, b\rangle_\sigma, & \forall\ b \in \mathcal{B}^-\mathbb{W}^{k,\ell}_{[p]}(\sigma), \quad \dim\sigma \geq \ell + p \end{cases}$$

are unisolvent for $C_{\iota^*\jmath^*_{[p]}}\mathcal{P}^-\mathbb{W}^{k,\ell}_{[p]}$. The resulting finite element space is $\iota^*\jmath^*_{[p]}$-conforming.

Remark on the shape function space
In any dimension, the lowest order space $C^{\iota^*\jmath^*}\mathcal{P}^-\mathbb{W}^{k,\ell}_{[p]}$ (with BGG symmetries) has constant shape function space $\mathbb{W}^{k,\ell}_{[p]}$ if $k \geq \ell + p - 1$. Only the case $k = \ell + p - 1$ (space right before zig-zag) appears in complexes.

- $p = 1$: (k, k)-forms, Regge, HHJ/TDNNS
- $p = 2$: $(k, k+1)$-forms, MCS
- $\cdots$

For $k < \ell + p - 1$: Koszul–type construction.

Distribution part: For $k \geq \ell + p - 1$,

$$D_{\iota^* \jmath^*_{[p]}} \widetilde{\mathbb{W}}^{k,\ell}_{[p]} := \mathrm{span}\{\omega \mapsto \langle \iota^* \jmath^*_{[p]}(\star\star)\omega, b\rangle_\sigma, \forall \; b \in \bigoplus_{s=0}^{p-1} \mathrm{Alt}^{n-k-s}(\sigma) \otimes \mathrm{Alt}^{k-\ell+s}(\sigma^\perp), \sigma \in \mathcal{T}^\circ_{n-k}\}.$$

BGG complex:
Now we can formally write down the BGG complex linking line ℓ and $\ell + p$.

$$\begin{aligned} 0 \longrightarrow C_{\iota^*\jmath^*_{[p]}} \mathcal{P}^- \mathbb{W}^{0,\ell}_{[p]} \longrightarrow C_{\iota^*\jmath^*_{[p]}} \mathcal{P}^- \mathbb{W}^{1,\ell}_{[p]} \longrightarrow \cdots \longrightarrow C_{\iota^*\jmath^*} \mathcal{P}^- \mathbb{W}^{\ell+p-1,\ell}_{[p]} \longrightarrow \\ \longrightarrow D_{\iota^*\jmath^*} \widetilde{\mathbb{W}}^{\ell+1,\ell+p}_{[p]} \longrightarrow \cdots \longrightarrow D_{\iota^*\jmath^*} \widetilde{\mathbb{W}}^{\ell+2,\ell+p}_{[p]} \longrightarrow D_{\iota^*\jmath^*} \widetilde{\mathbb{W}}^{n,\ell+p}_{[p]} \longrightarrow 0, \end{aligned}$$

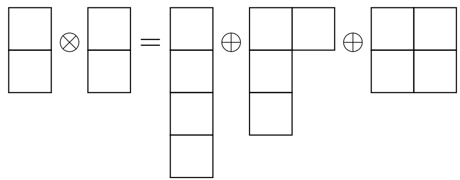
Dimension count (Euler characteristics) works.

SOME EXAMPLES IN 4D

- distributional curvature, 4D elasticity complex

$$0 \longrightarrow \mathbf{Lag}\otimes\mathbb{V} \xrightarrow{\text{sym grad}} \mathbf{Reg} \xrightarrow{\text{inc 4}} \bigoplus_{f\in\mathcal{T}_2^\circ}\delta_{mm}(f) \xrightarrow{\text{curl}} \bigoplus_{e\in\mathcal{T}_1^\circ}\delta_{mt}(e) \xrightarrow{\text{div}} \bigoplus_{v\in\mathcal{T}_0^\circ}\delta(v)\otimes\mathbb{K} \longrightarrow 0.$$

  $\bigoplus_{f\in\mathcal{T}_2^\circ}\delta_{mm}(f)$: combination of functionals against (2,2)-forms on 2-cells.
  (reminiscence of sectional curvature).
- curvature tensor (2,2)-form, dual of 4D elasticity complex
  tensor decomposition from the BGG perspective
  2-form valued 2-forms = 4-forms + 'skew (2, 2)-form' + Riemann-like tensor (algebraic curvature)



$$6 * 6 \quad = 1 \quad + \quad 15 \quad + \quad 20$$

$$\mathrm{Alt}^{2,2} = \mathcal{S}^{2,2}_{\dagger,[2]}(\mathrm{Alt}^{2,2}) \oplus \left[\ker(\mathcal{S}^{2,2}_{\dagger,[2]}) \cap \mathcal{S}^{2,2}_{\dagger}(\mathrm{Alt}^{2,2})\right] \oplus \ker(\mathcal{S}^{2,2}_{\dagger})$$

Projections to the 1st and 3rd components: Bianchi symmetrization, Kulkarni–Nomizu operator

$$0 \longrightarrow \mathbf{Lag}\otimes\mathbb{K} \longrightarrow C_{\iota^*\jmath^*}\mathcal{P}^-\mathbb{W}^{1,2} \longrightarrow C_{\iota^*\iota^*}\mathcal{P}^-\mathbb{W}^{2,2} \longrightarrow \bigoplus_{e\in\mathcal{T}_1^\circ}\delta_{tt}(e) \longrightarrow \bigoplus_{v\in\mathcal{T}_0^\circ}\delta(v)\otimes\mathbb{V} \longrightarrow 0.$$

SUMMARY

What have been done:

- Discretization of the entire BGG *diagram*; $\mathcal{P}_r^-\Lambda^{k,\ell}$, $\mathcal{P}_r\Lambda^{k,\ell}$, for any r, k, ℓ, dimension n.
  Corresponding BGG complexes with symmetries encoded in BGG diagrams.

Still open:

- Cohomology (rigorously proved in $\leq$ 3D, lowest order; dimension count holds for any dimention).
- Conformal-type complexes (transverse-traceless gauge in gravitational waves etc.).

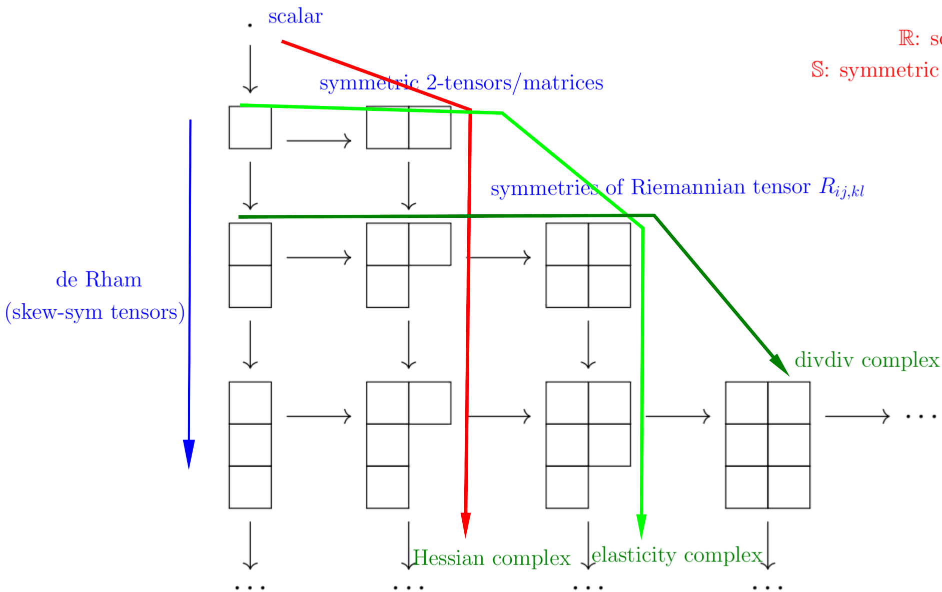
Practical benefits of distributional-FE-based methods (thanks to J.Schöberl):

- invariance under pullbacks
  useful for assembling FE matrices and mapping to curved manifolds such as shells (intrinsic finite elements).
- parameter-robustness, anisotropic problems ('Model Reduction via Discretization', J.Schöberl, IMA Structure-preserving workshop, 2014)

Young tableaux provides another way to parametrize FEs

$\mathbb{R}$: scalar field $\mathbb{V}$: vectors

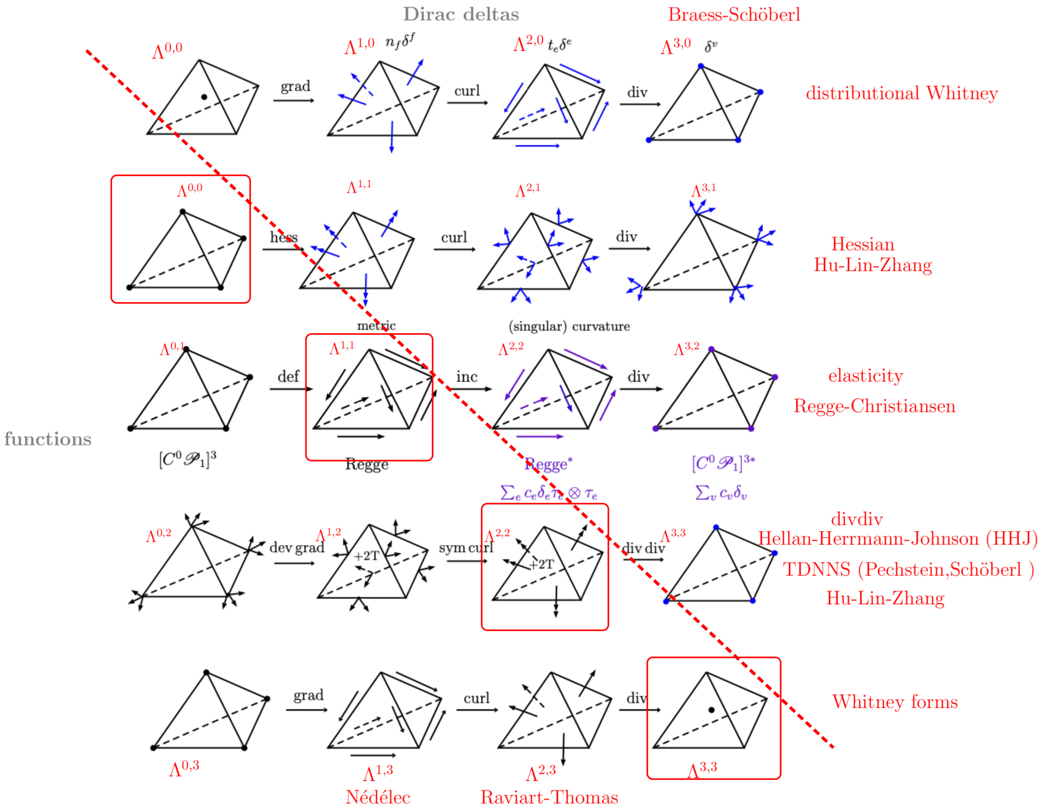
$\mathbb{S}$: symmetric matrix field $\mathbb{T}$: tracefree matrix field



$$
\begin{tikzcd}
\mathbb{R} \arrow[d, "\mathrm{grad}"] & & & \\
\mathbb{V} \arrow[r, "\mathrm{sym\,grad}"] \arrow[d, "\mathrm{curl}"] & \mathbb{S} \arrow[d, "\mathrm{curl}"] & & \\
\mathbb{V} \arrow[r, "\mathrm{dev\,grad}"] \arrow[d, "\mathrm{div}"] & \mathbb{T} \arrow[r, "\mathrm{sym\,curl}"] \arrow[d, "\mathrm{div}"] & \mathbb{S} \arrow[d, "\mathrm{div}"] & \\
\mathbb{R} \arrow[r, "\mathrm{grad}"] & \mathbb{V} \arrow[r, "\mathrm{curl}"] & \mathbb{V} \arrow[r, "\mathrm{div}"] & \mathbb{R}
\end{tikzcd}
$$

Peter Olver, 'Differential hyperforms' 1982.

Discretization: ongoing work with Jay Gopalakrishnan, Joachim Schöberl.



Dirac deltas
Braess-Schöberl
$\Lambda^{0,0}$
$\Lambda^{1,0}$ $n_f\delta^f$
$\Lambda^{2,0}$ $t_e\delta^e$
$\Lambda^{3,0}$ δ^v
grad
curl
div
distributional Whitney
$\Lambda^{0,0}$
$\Lambda^{1,1}$
$\Lambda^{2,1}$
$\Lambda^{3,1}$
hess
curl
div
Hessian
Hu-Lin-Zhang
metric
(singular) curvature
$\Lambda^{0,1}$
$\Lambda^{1,1}$
$\Lambda^{2,2}$
$\Lambda^{3,2}$
def
inc
div
elasticity
Regge-Christiansen
functions
$[C^0\mathscr{P}_1]^3$
Regge
Regge*
$[C^0\mathscr{P}_1]^{3*}$
$\sum_e c_e\delta_e\tau_e \otimes \tau_e$
$\sum_v c_v\delta_v$
$\Lambda^{0,2}$
$\Lambda^{1,2}$
$\Lambda^{2,2}$
$\Lambda^{3,3}$
dev grad
sym curl
div div
+2T
+2T
divdiv
Hellan-Herrmann-Johnson (HHJ)
TDNNS (Pechstein,Schöberl)
Hu-Lin-Zhang
grad
curl
div
$\Lambda^{0,3}$
$\Lambda^{1,3}$
$\Lambda^{2,3}$
$\Lambda^{3,3}$
Nédélec
Raviart-Thomas
Whitney forms

Thank you!