

FEEC FOR COMPUTATIONAL TOPOLOGICAL (MAGNETO)HYDRODYNAMICS

LONG TERM DYNAMICS AND COMPUTATION OF FLUIDS / PLASMA

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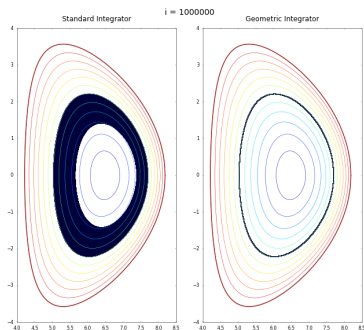


Aknowledgement

Royal Society University Research Fellowship (URF\R1\221398)

ERC starting grant (101164551): **GeoFEM** (Geometric Finite Element Methods)

“A fundamental principle in the study of computational methods is that the *essential characteristics of the original problem* should be *preserved as much as possible after discretization.*”



Max-Planck Institute for Plasma Physics

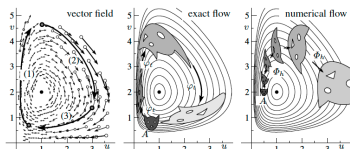
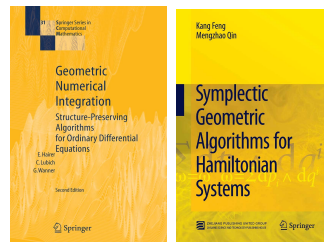


Fig. 1.1. Vector field, exact flow, and numerical flow for the Lotka-Volterra model (1.1)



A different way of thinking

Local analysis	Global analysis
$t \in [0, T]$, $\Delta t, h \rightarrow 0$	Long-time asymptotics
Finite-time converg.	stability & converg. (e.g., Discrete KAM)
Approximation error	Dynamical-system properties
Local accuracy	Qualitative behaviour
Example: preserve canonical symplectic form	

$$\omega = \sum_{i=1}^n dq_i \wedge dp_i, \quad \Phi_h^* \omega = \omega.$$

FLUID AND PLASMA SYSTEMS

Various conserved quantities:

- ▶ energy,
- ▶ mass $\text{div } \mathbf{u} = 0$ (for incompressible flows),
- ▶ charge $\text{div } \mathbf{j} = \rho$,
- ▶ helicity $\int \mathbf{A} \cdot \mathbf{B} \, dx$,
- ▶ enstrophy $\int |\text{curl } \mathbf{u}|^2 \, dx$,
- ▶ ...

A structure-preserving faith: preserve them as much as possible.

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This talk: **long term dynamics and computation of fluids and plasma**

- ▶ **relaxation** : *Does plasma system evolve to a stationary state?*
corona heating, MHD equilibrium (stellarator, tokamak) etc.
- ▶ **dynamo** : *Does the magnetic field exponentially grow?*
generation of magnetic fields in earth and sun etc.

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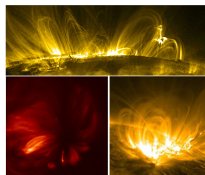
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generation of magnetic fields in earth and sun etc.

and limits of preserving invariants for fluids?

MOTIVATION: STRUCTURE-PRESERVING DISCRETISATION

Fundamental question in plasma physics: given initial data, what does the system evolve to?
heating of solar corona, plasma equilibria (magnetic configurations) etc.



Magneto-friction (simplified MHD) :

$$\begin{aligned}\mathbf{B}_t - \nabla \times (\mathbf{u} \times \mathbf{B}) &= 0, \\ \mathbf{j} &= \nabla \times \mathbf{B}, \\ \mathbf{u} &= \tau \mathbf{j} \times \mathbf{B}.\end{aligned}$$

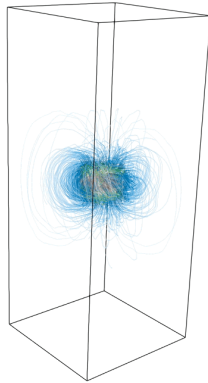
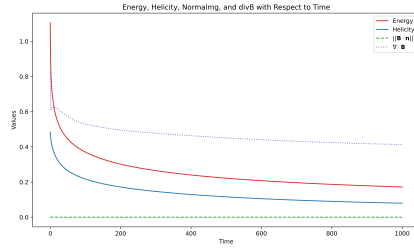
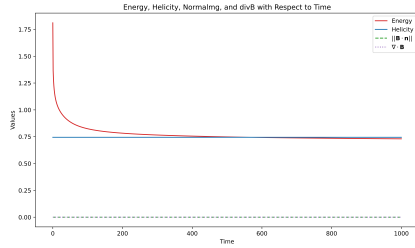
Energy decay

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{B}\|^2 = -\tau \|\mathbf{B} \times \mathbf{j}\|^2.$$

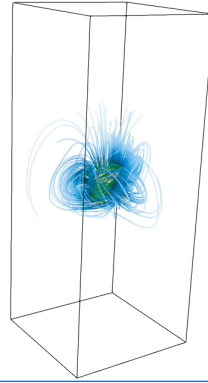
Helicity conservation

$$\frac{d}{dt} \mathcal{H}_m = 0, \quad \text{with } \mathcal{H}_m := \int \mathbf{A} \cdot \mathbf{B} \, dx, \quad \mathbf{B} = \nabla \times \mathbf{A}.$$

HOPF FIBRATION

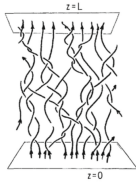
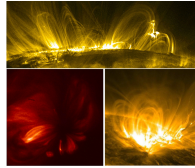


Helicity-preserving scheme



Plain FEM (even standard FEED)

IDEAL MAGNETIC RELAXATION

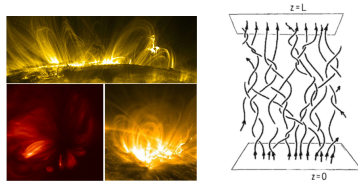


Eugene Parker

Parker hypothesis (Still Open)

For “almost any initial data”, *the magnetic field develops tangential discontinuities (current sheet) during the relaxation to static equilibrium.*

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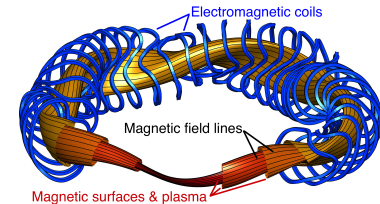
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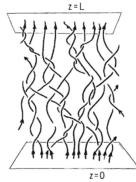
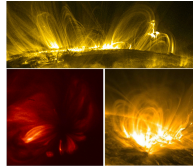
Design of magnetic-confinement fusion devices (stellarator/Tokamak):

force-free state $\nabla p = \mathbf{j} \times \mathbf{B}$, $\nabla \cdot \mathbf{B} = 0$.

computing MHD equilibria



IDEAL MAGNETIC RELAXATION



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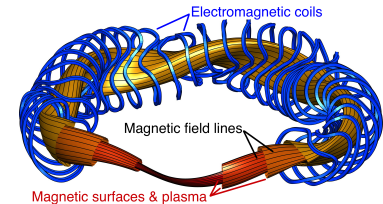
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computing MHD equilibria



How confident are we in what we compute?

Relaxation, Self-organization

When energy decays,
what remains?

Topology versus dissipation

Magnetohydrodynamics (MHD): macroscopic description of plasma, an incompressible model

$$\begin{aligned}
 \partial_t \mathbf{u} - \mathbf{u} \times (\nabla \times \mathbf{u}) - R_e^{-1} \Delta \mathbf{u} - s \mathbf{j} \times \mathbf{B} + \nabla P &= \mathbf{f} && \text{momentum equation,} \\
 \mathbf{j} - \nabla \times \mathbf{B} &= \mathbf{0} && \text{Ampere's law,} \\
 \partial_t \mathbf{B} + \nabla \times \mathbf{E} &= \mathbf{0} && \text{Faraday's law,} \\
 R_m^{-1} \mathbf{j} - (\mathbf{E} + \mathbf{u} \times \mathbf{B}) &= \mathbf{0} && \text{Ohm's law,} \\
 \nabla \cdot \mathbf{B} &= 0 && \text{Gauss law,} \\
 \nabla \cdot \mathbf{u} &= 0, &&
 \end{aligned}$$

initial conditions $\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x})$, $\mathbf{B}(\mathbf{x}, 0) = \mathbf{B}_0(\mathbf{x})$,

boundary conditions on $\partial\Omega$: $\mathbf{u} = \mathbf{0}$, $\mathbf{B} \cdot \mathbf{n} = 0$, $\mathbf{E} \times \mathbf{n} = \mathbf{0}$.

Three nonlinear terms:

fluid advection $-\mathbf{u} \times (\nabla \times \mathbf{u})$ (in the vorticity form)

Lorentz force $-s \mathbf{j} \times \mathbf{B}$

magnetic advection $-\nabla \times (\mathbf{u} \times \mathbf{B})$

For relaxation, we are interested in zero magnetic diffusion, nonzero fluid diffusion ($R_m = \infty$, $R_e < \infty$).

ENERGY STRUCTURES OF MHD

Energy dissipation or conservation:

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{u}\|_0^2 + \frac{S}{2} \frac{d}{dt} \|\mathbf{B}\|_0^2 + R_e^{-1} \|\nabla \mathbf{u}\|_0^2 + SR_m^{-1} \|\mathbf{j}\|_0^2 = (\mathbf{f}, \mathbf{u}),$$

and hence

$$\begin{aligned} \max_{0 \leq t \leq T} \left(\|\mathbf{u}\|_0^2 + S \|\mathbf{B}\|_0^2 \right) + R_e^{-1} \int_0^T \|\nabla \mathbf{u}\|_0^2 d\tau + 2SR_m^{-1} \int_0^T \|\mathbf{j}\|_0^2 d\tau \\ \leq \|\mathbf{u}_0\|_0^2 + S \|\mathbf{B}_0\|_0^2 + R_e \int_0^T \|\mathbf{f}\|_{-1}^2 d\tau. \end{aligned}$$

With $\mathbf{f} = 0$, $R_m^{-1} = 0$, total energy is non-increasing. However, some key information is **not clear**:

- ▶ whether the total energy decays to zero?
- ▶ how does total energy split into the fluid part ($\|\mathbf{u}\|^2$) + magnetic part ($S\|\mathbf{B}\|^2$)?

HELICITY: FINE STRUCTURES

Magnetic helicity: for any potential $\mathbf{A}$ satisfying $\nabla \times \mathbf{A} = \mathbf{B}$,

$$\mathcal{H}_m := \int_{\Omega} \mathbf{A} \cdot \mathbf{B} \, dx$$

Idea started from Helmholtz & Kelvin. MHD: Woltjer's invariant, ideal fluid: Moffatt (giving the name).

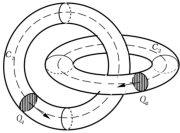


Figure: linking/knottedness of $\mathbf{B}$.

$\mathcal{H}_{\xi} = 2\ell(C_1, C_2)Q_1 \cdot Q_2$. $\ell = 1$: Gauss linking number, Q_i : flux

Helicity = averaging asymptotic linking number (V.I. Arnold)

Cross helicity:

$$\mathcal{H}_c := \int_{\Omega} \mathbf{u} \cdot \mathbf{B} \, dx$$

linking of vorticity and magnetic fields

A TOPOLOGICAL MECHANISM

Arnold inequality (V.I. Arnold 1974): helicity provides lower bound for energy

$$\left| \int \mathbf{A} \cdot \mathbf{B} \, dx \right| \leq C \int |\mathbf{B}|^2 \, dx$$

Proof. Cauchy-Schwarz $|\int \mathbf{A} \cdot \mathbf{B} \, dx| \leq \|\mathbf{A}\|_{L^2} \|\mathbf{B}\|_{L^2}$ + Poincaré inequality $\|\mathbf{A}\|_{L^2} \leq C \|\nabla \times \mathbf{A}\|_{L^2}$.



Vladimir I. Arnold

Differential form point of view: $\mathbf{A}$: 1-form, $\mathbf{B}$: 2-form

$$\int \mathbf{A} \wedge \mathbf{B}$$

$$\leq C$$

$$\int \mathbf{B} \wedge * \mathbf{B}$$

Helicity, Topology

Energy, Geometry

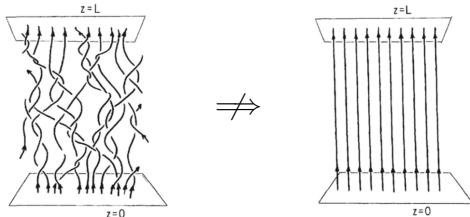


Fig: Pontin, Hornig, Living Rev. Sol. Phys. 2020.

knots are topological barriers that prevent energy from dissipation

MAGNETIC HELICITY CONSERVATION

For Ideal MHD ($R_m = \infty$), $\frac{d}{dt} \int_{\Omega} \mathbf{A} \cdot \mathbf{B} \, dx = 0$.

One-line proof

$$\mathbf{B}_t = \nabla \times (\mathbf{u} \times \mathbf{B})$$

$$\frac{d}{dt} \int \mathbf{A} \cdot \mathbf{B} = 2 \int (\nabla \times \mathbf{A}) \cdot (\mathbf{u} \times \mathbf{B}) = 2 \int \mathbf{u} \cdot (\mathbf{B} \times \mathbf{B}) = 0.$$

Proof holds for any $\mathbf{u}$!

Why it matters

- ▶ Energy may decay, but helicity (topology) constrains the final state (lower bound).
- ▶ A discretization that destroys helicity may converge to the wrong equilibrium.

NUMERICAL TEST: HOPF FIBRATION

$$\mathbf{B}_0 = \frac{4\sqrt{a}}{\pi(1+r^2)^3}(2y(y-xz), -2(x+yz), (-1+x^2+y^2-z^2))$$

Every single field line of this field is a perfect circle, and every single field line is linked with every other one.
c.f. Smiet, C.B., Candelaresi, S. and Bouwmeester, D., 2017. Ideal relaxation of the Hopf fibration. Physics of Plasmas, 24(7).

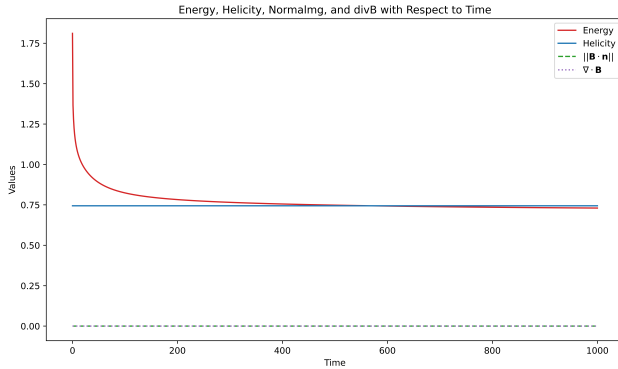


Figure. Helicity-preserving scheme

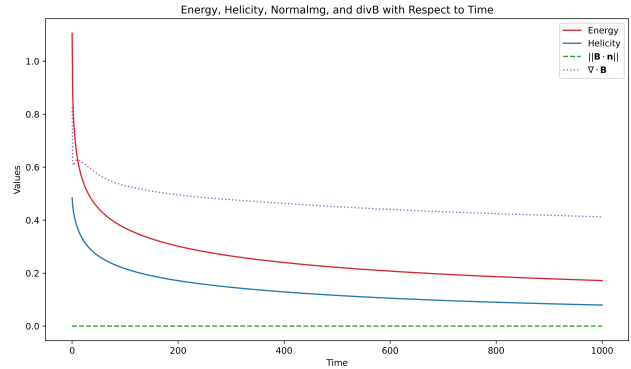


Figure. CG scheme (non-preserving)

$\tau = 10$, $dt = 1$ and $T = 1000$.

SPURIOUS NUMERICAL RECONNECTION

Helicity-preserving:

Plain methods:

STRUCTURE-PRESERVING DISCRETIZATIONS FOR MHD

Existing structure-preserving methods

- ▶ **Energy conservation:** Armero–Simo (1996), ...
- ▶ $\nabla \cdot \mathbf{B} = 0$: Brackbill–Barnes (1980), Hu–Ma–Xu (2017), Hiptmair–Mao–Zheng (2018), ...
- ▶ **Charge conservation:** Li–Ni–Zheng (2019)

Magnetic helicity preservation

Rebholz (2007)	Navier-Stokes equations
Liu–Wang (2004)	axisymmetric MHD
Kraus–Maj (2017)	ideal MHD (DEC / variational integrator)
Sullivan (2018)	lattice hydrodynamics / discrete topology

Hu–Lee–Xu (2021)

FEEC

Related developments

- ▶ Gawlik–Gay-Balmaz (2022)
- ▶ Laakmann–Hu–Farrell (2023): Hall MHD
- ▶ Zhang–Palha–Brugnoli–Toshniwal–Gerritsma (2024)
- ▶ **Matrix hydrodynamics:** Zeitlin, Modlin, Roop, Viviani...

Numerics in this talk: mixed finite element / FEEC approach

ROLE OF COMPLEX: EXAMPLE WITH GAUSS' LAW

Continuous Maxwell equations

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0 \quad \implies \quad \partial_t (\nabla \cdot \mathbf{B}) = 0.$$

After Galerkin discretization

$$\partial_t \mathbf{B}_h + \mathbb{P}(\nabla \times \mathbf{E}_h) = 0, \quad \mathbf{B}_h \in Z_h, \mathbf{E}_h \in Y_h.$$

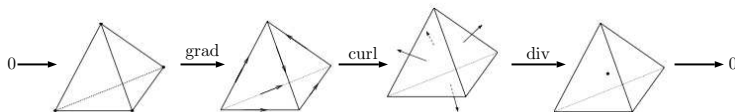
$$\partial_t (\nabla \cdot \mathbf{B}_h) = -\nabla \cdot \mathbb{P}(\nabla \times \mathbf{E}_h) \neq 0.$$

Where does the error come from?

$$\mathbb{P} \neq I \quad \iff \quad \nabla \times Y_h \not\subset Z_h.$$

Compatible finite elements

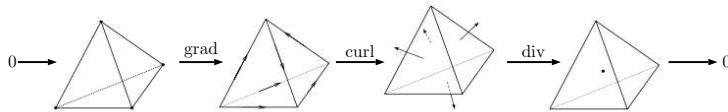
$$\nabla \times Y_h \subset Z_h$$



Variables should be discretized according to a complex.

CANONICAL FINITE ELEMENTS FOR THE DE RHAM COMPLEX

$$0 \longrightarrow \Lambda_h^0 \xrightarrow{\text{grad}} \Lambda_h^1 \xrightarrow{\text{curl}} \Lambda_h^2 \xrightarrow{\text{div}} \Lambda_h^3 \longrightarrow 0$$



Canonical finite elements form an exact discrete complex

- ▶ Raviart–Thomas (1977)
- ▶ Nédélec (1980)
- ▶ Bossavit (1988): differential forms and complexes
- ▶ Hiptmair (1999), Arnold–Falk–Winther (2006):
Finite Element Exterior Calculus (FEEC)

Periodic table of finite elements

Periodic Table of the Finite Elements

The periodic table of finite elements is a grid of 16 columns and 4 rows. Each cell contains a small diagram of a finite element and its corresponding mathematical representation. The elements are organized by their degree and type, with a legend at the bottom right.

Why it matters

Compatible finite elements preserve Gauss-type constraints and the de Rham structure.

This is the language used later for MHD, helicity, and topology.

STRUCTURE-PRESERVING ALGORITHM FOR MAGNETO-FRICTION

Core idea

Use compatible finite elements in a de Rham complex + structure-preserving projections.

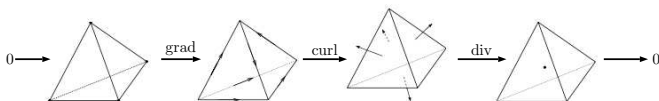
Semi-discrete scheme

$$\begin{aligned} \mathbf{B}_t + \nabla \times \mathbf{E} &= 0, \\ \mathbf{E} &= -Q_h^{\text{curl}}(\mathbf{u} \times \mathbf{H}), \\ \mathbf{u} &= \tau Q_h^{\text{curl}}(\mathbf{j} \times \mathbf{H}), \\ \mathbf{j} &= \nabla_h \times \mathbf{B}, \\ \mathbf{H} &= Q_h^{\text{curl}} \mathbf{B}. \end{aligned}$$

Compatible spaces

$$\mathbf{E}, \mathbf{j}, \mathbf{H} \in H_0^h(\text{curl}), \quad \mathbf{B}, \mathbf{u} \in H_0^h(\text{div}).$$

$$\text{curl } H_0^h(\text{curl}) \subset H_0^h(\text{div}).$$



Why the projection?

$$(\mathbf{u} \times \mathbf{B}, \mathbf{B}) = 0.$$

$$(\mathbf{u} \times \mathbf{B}_h, Q_h^{\text{curl}} \mathbf{B}_h) \neq 0.$$

$$\mathbf{H}_h := Q_h^{\text{curl}} \mathbf{B}_h \implies (\mathbf{u} \times \mathbf{H}_h, \mathbf{H}_h) = 0.$$

Main properties

- $\nabla \cdot \mathbf{B}_h = 0$ exactly;
- energy dissipates;
- magnetic / cross helicity preserved.

A LAGRANGE MULTIPLIER APPROACH

Can we instead preserve

$$H(\Omega) = \int_{\Omega} \mathbf{A} \cdot \mathbf{B} \, dx$$

using only one scalar Lagrange multiplier?

Continuous formulation

$$\partial_t \mathbf{A} + \mathbf{E} + \lambda_{\mathcal{H}} \mathbf{B} + \lambda_{\mathcal{E}} \mathbf{j} = 0,$$

$$\mathbf{B} = \nabla \times \mathbf{A},$$

$$\mathbf{u} - \tau \mathbf{j} \times \mathbf{B} = 0,$$

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = 0,$$

$$\mathbf{j} = \nabla \times \mathbf{B},$$

$$\frac{d}{dt} \mathcal{E} = -2\tau \|\mathbf{j} \times \mathbf{B}\|^2,$$

$$\frac{d}{dt} \mathcal{H}(\Omega) = 0.$$

Unknowns

$$(\mathbf{A}, \mathbf{B}, \mathbf{E}, \mathbf{j}, \mathbf{u})$$

$$+ (\lambda_H, \lambda_E) \in \mathbb{R}^2.$$

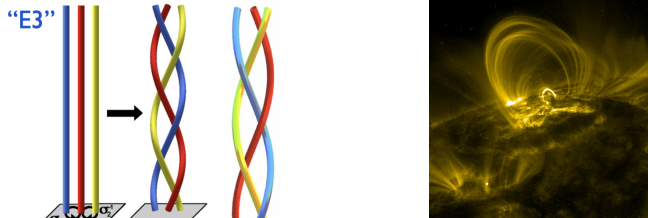
One scalar constraint

instead of

one auxiliary field.

Woltjer's variational principle: $\frac{\delta \mathcal{E}}{\delta \mathbf{A}} = 2\mathbf{j}$, $\frac{\delta \mathcal{H}}{\delta \mathbf{A}} = 2\mathbf{B}$

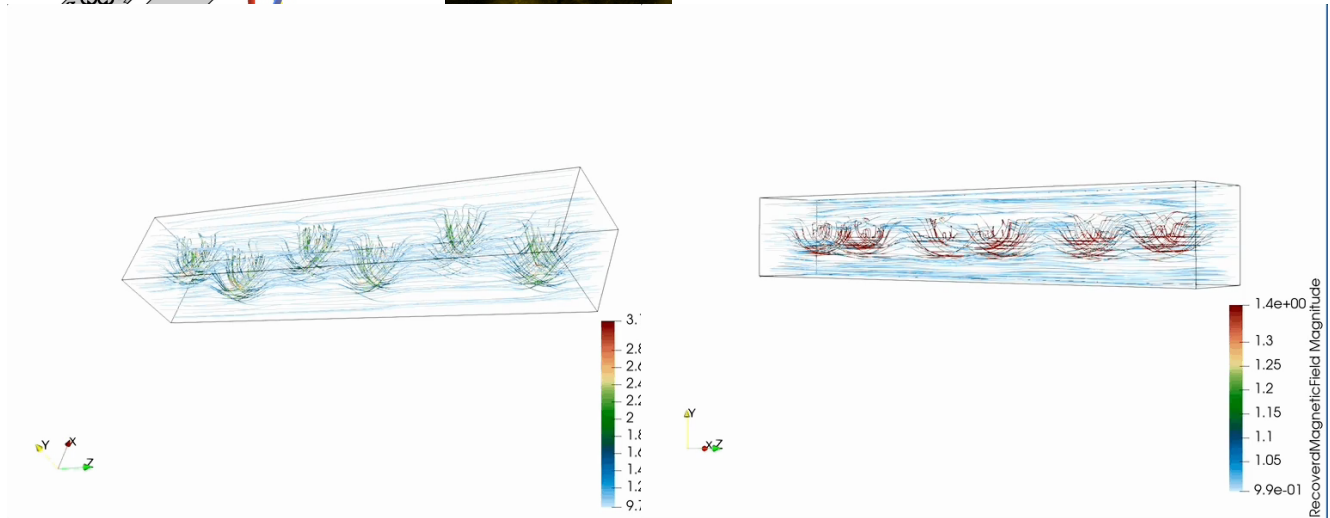
LOCAL VS. GLOBAL CONSERVATION



nontrivial topology with zero global helicity.

- ▶ magnetic braids,
- ▶ coronal loops.

Pontin–Wilmot-Smith–Hornig–Yeates



Left: helicity-preserving algorithm **Right:** preserving integrated $\int \mathbf{A} \cdot \mathbf{B} dx$ by Lagrange multiplier.

preserving *global* $\int \mathbf{A} \cdot \mathbf{B} dx$ not enough! local structures matter.

Broader question: evolve *table of knots* (not theoretically guaranteed for numerics yet).

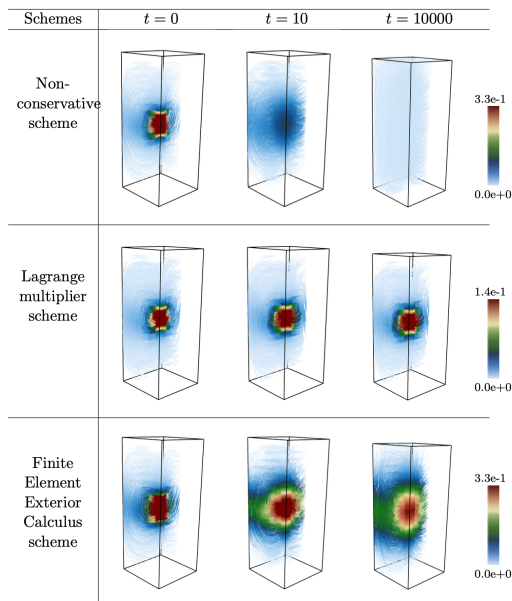
To preserve,
or not to preserve?

That is the question.

Invariant preservation has a limit.

In finite dimensions (ODEs), preserve invariants as much as possible. This is subtle in infinite dimensions. Conservation breaks down in the presence of diffusion (particle collisions) or singularity.

True physics always has diffusion $\Rightarrow$ reconnection.



- Selective decay:
helicity decays much slower than energy
more robust invariant, nearly conserved
- Taylor's conjecture:
global helicity conserved,
local helicity not

**Different topology constraints
 $\Rightarrow$ different equilibria.**

Which one corresponds to real physics?

(more stories, leading to discussions on turbulence...)

Magnetic knots (Hopf fibration): evolution under different topological constraints

CONVERGENCE

Algorithms converge well for *smooth true solutions*.

Theorem 1 (L. Beirão da Veiga, KH, L. Mascotto 2024)

Consider sequences $\{\mathcal{T}_h\}$ of shape-regular, quasi-uniform meshes. Let the true solution be **sufficiently smooth**. Then, there exists a positive constant C independent of h such that, for all t in $(0, T]$,

$$\|\mathbf{e}_h^u(t)\|^2 + \|\mathbf{e}_h^B(t)\|^2 + \int_0^t \|\operatorname{curl} \mathbf{e}_h^u(s)\|^2 ds + \int_0^t \|\mathbf{e}_h^j(s)\|^2 ds \leq C(\|\mathbf{e}_h^u(0)\|^2 + \|\mathbf{e}_h^B(0)\|^2 + h^{2(k+1)}).$$

The constant C includes regularity terms of the numerical solution, the shape-regularity parameter of the mesh, and the polynomial degree k .

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The constant C includes regularity terms of the numerical solution, the shape-regularity parameter of the mesh, and the polynomial degree k .

Further question

What if the true solution is **nonsmooth**?

Onsager's conjecture; energy/helicity conservation may fail. But most FE preserves energy by definition. $O(1)$ error!

invariants-preservation for fluids has a limit!

LIMIT OF INVARIANT-PRESERVATION

in numerical simulations. we can use an indirect approach, since it exploits the connection between singular behaviour and anomalous energy dissipation according to Onsager's conjecture. A decisive point is that this technique requires suitable discretisation schemes that remain robust in the presence of singularities and provide mechanisms of dissipation in case no viscous dissipation is present, which is a challenge in itself. Then, the idea is that observing energy-dissipating behaviour for a sequence of mesh refinement levels provides insight into the physical dissipation behaviour of the problem under investigation.

— Fehn, N., Kronbichler, M., Munch, P., & Wall, W. A. (2022). *Numerical evidence of anomalous energy dissipation in incompressible Euler flows: towards grid-converged results for the inviscid Taylor–Green problem*. Journal of Fluid Mechanics, 932, A40.

Motivation: detecting NS singularity

- ▶ direct approach: blow-up of $\mathbf{u}$, $\boldsymbol{\omega}$ etc. (e.g., Hou-Luo 2014)
- ▶ indirect approach: **energy dissipation for inviscid flows**

Enforcing energy conservation has to be wrong.

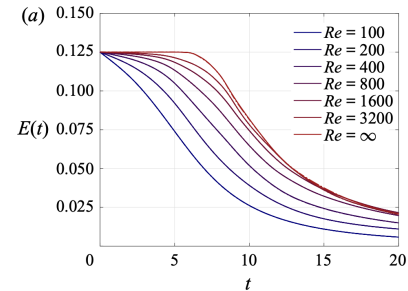


Figure 1. Temporal evolution of kinetic energy

high-order projection methods + high-order DG

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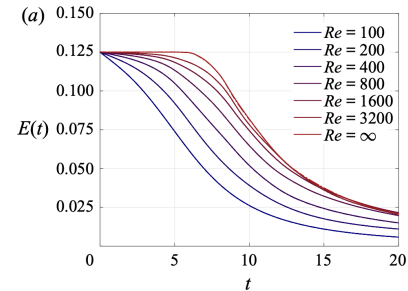


Figure 1. Temporal evolution of kinetic energy

high-order projection methods + high-order DG

heuristic argument

Helicity is also more robust than energy in weak regularity regime, since $\int \mathbf{A} \cdot \mathbf{B}$ has fewer derivatives than $\int \mathbf{B} \cdot \mathbf{B}$.

Dynamo

Can a magnetic field
grow forever?

Long-time instability

Dynamo theory

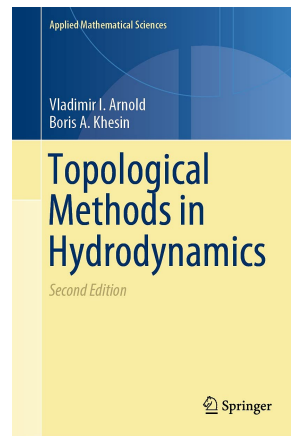
The dynamo mechanism explains how conducting fluids generate and sustain magnetic fields in stars, planets and galaxies.

Fast (kinetic) dynamo:

The magnetic field grows exponentially under the induction equation

$$\partial_t \mathbf{B} - \nabla \times (\mathbf{u} \times \mathbf{B}) + R_m^{-1} \nabla \times \nabla \times \mathbf{B} = 0.$$

Given a velocity field $\mathbf{u}$, Does the magnetic field grow exponentially?



Arnold & Khesin
*Topological Methods in
Hydrodynamics*

CAN COMPUTATION ANSWER THE DYNAMO QUESTION?

... It is **still unknown** whether this field (ABC flow) is a fast kinematic dynamo, e.g., whether an exponentially growing mode of B survives as $R_m \rightarrow \infty$.

...

Numerically, the kinematic fast dynamo problem is the **first eigenvalue problem** for matrices of the order of many million, even for reasonable Reynolds numbers (of the order of hundreds). The physically meaningful magnetic Reynolds numbers R_m are of order of magnitude 10^8 . **The corresponding matrices are (and will remain) beyond the reach of any computer.**

— *Topological Methods in Hydrodynamics*, Arnold & Khesin (1998)

Standard numerical approach

Study the leading eigenvalue of the magnetic advection–diffusion operator

$$\nabla \times (\mathbf{u} \times \mathbf{B}) - R_m^{-1} \nabla \times \nabla \times \mathbf{B} = \lambda \mathbf{B}.$$

$$\text{Fast dynamo} \quad \Longleftrightarrow \quad \Re(\lambda) > 0$$

V.I. Arnold & Korkina (1983): Galerkin computations up to $R_m \leq 19$.

MHD VIA DIFFERENTIAL FORMS

$$\underbrace{-\nabla \times (\mathbf{u} \times \mathbf{B})}_{\substack{\text{Lie derivative} \\ \text{advection}}} + \underbrace{-R_m^{-1} \nabla \times \nabla \times \mathbf{B}}_{\substack{\text{Hodge Laplacian} \\ \text{diffusion}}} = \lambda \mathbf{B}.$$

Lie derivative

For a vector field $\mathbf{u}$ and a differential form ω ,

$$L_{\mathbf{u}}\omega = \lim_{\tau \rightarrow 0} \frac{\Phi_{\tau}^* \omega - \omega}{\tau},$$

where Φ is the flow generated by $\mathbf{u}$.

Hodge Laplacian

$$\Delta_{\text{HL}} = d\delta + \delta d.$$

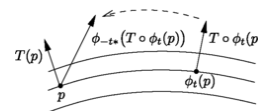
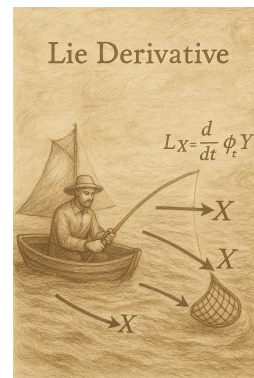
Cartan's Magic Formula:

$$L_{\beta}^k = d^{k-1} i_{\beta}^k + i_{\beta}^{k+1} d^k$$

where $i_{\beta}^k : \Lambda^k \rightarrow \Lambda^{k-1}$: contraction.

Fisherman derivative

Differentiate while moving with the flow.



ONE GEOMETRIC OPERATOR, FOUR CLASSICAL PDEs

$$\underbrace{L_u w}_{\substack{\text{Lie derivative} \\ \text{advection}}} + \underbrace{\Delta_{HL} w}_{\substack{\text{Hodge Laplacian} \\ \text{diffusion}}} = f$$

0-forms	$\beta \cdot \nabla u - \Delta u = f$	scalar advection–diffusion
1-forms	$\text{grad}(\beta \cdot A) - \beta \times (\text{curl } A) + (-\text{grad div} + \text{curl curl})A = f$	magnetic potential
2-forms	$-\text{curl}(\beta \times B) + \text{curl curl } B = f$	magnetic induction
n -forms	$\text{div}(u\beta) - \Delta u = f$	Fokker–Planck

Differential forms provide one geometric framework for a family of advection–diffusion equations.

CONVERGENCE OF ADVECTION–DIFFUSION EIGENVALUE PROBLEMS

Advection–diffusion eigenproblem

$$R_m^{-1}(\operatorname{curl} \mathbf{B}, \operatorname{curl} \mathbf{C}) - (\mathbf{u} \times \mathbf{B}, \operatorname{curl} \mathbf{C}) = \lambda(\mathbf{B}, \mathbf{C}), \quad \forall \mathbf{C} \in H(\operatorname{curl}).$$

Babuška–Osborn spectral approximation

Assume the solution operators

$$T : X \rightarrow X, \quad T_h : X_h \rightarrow X_h$$

are compact (with T_h of finite rank). Then

$$\|T - T_h\| \longrightarrow 0 \quad \implies \quad \text{spectral convergence.}$$

Key analytical step (KH–Liang–Zerbinati)

For sufficiently smooth velocity fields ($\mathbf{u} \in W^{1,\infty}$), compact embedding

$$V_0 := T(L^2) \hookrightarrow H(\operatorname{curl}).$$

Further implies operator convergence.

Rayleigh quotient (min–max) fails.

BACK TO THE FIRST ASSUMPTION...

*“Numerically, the kinematic fast dynamo problem is the **first eigenvalue problem** for matrices of many millions. The corresponding matrices are (and will remain) beyond the reach of any computer.”*

— Topological Methods in Hydrodynamics, Arnold & Khesin (2021)

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*“**Eigenvalues** might be meaningful in theory, but they cannot always be trusted on a computer. Look beyond eigenvalues to **pseudospectra**.”*

— Trefethen & Embree (2005)

Eigenvalue analysis is often **misleading** for **non-normal operators** ($AA^* \neq A^*A$).

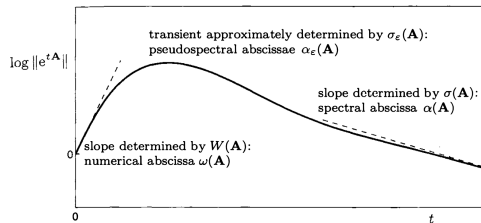
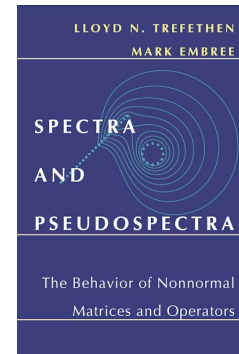


Figure 14.1: Initial, transient, and asymptotic behavior of $\|e^{tA}\|$ for a nonnormal matrix or operator A .



STRUCTURE-PRESERVING FE ALSO COMPUTES PSEUDOSPECTRA

Pseudospectrum

$$\sigma_\varepsilon(A) = \left\{ z \in \mathbb{C} : \|(z - A)^{-1}\| > \varepsilon^{-1} \right\} = \{ z \in \mathbb{C} : \lambda_{\min}(z - A) < \varepsilon \}.$$

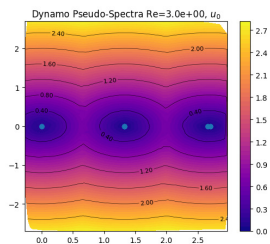
$\lambda_{\min}$: smallest singular value.

Theorem (Boffi–KH–Liang–Zerbinati)

If $\|T_h - T\|_{\mathcal{L}(X,X)} \rightarrow 0$, then for every $0 < \delta < \varepsilon$,

$$\sigma_\delta(T_h) \subset \sigma_\varepsilon(T), \quad \sigma_\delta(T) \subset \sigma_\varepsilon(T_h),$$

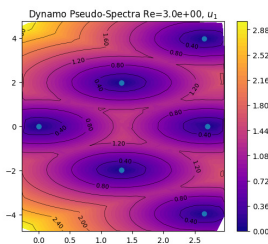
for all sufficiently fine meshes.



$\mathbf{u}_0 = 0$

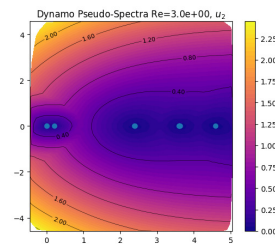
$\mathbf{u}_1 = (1, 1)$

Hodge Laplacian



$\mathbf{u}_2 = (2 \cos 2x \sin 2y, 2 \sin 2x \cos 2y)$

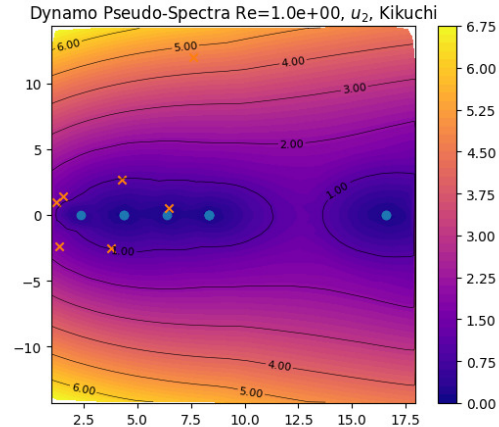
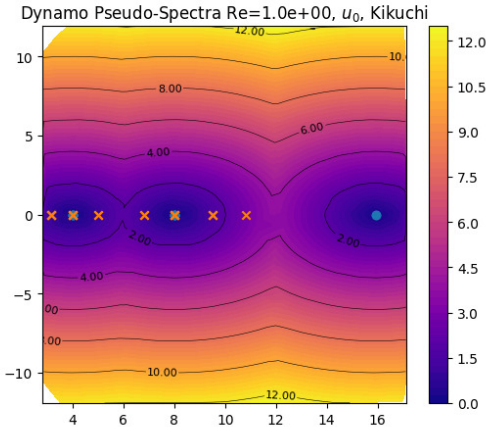
non-potential



potential

LAGRANGE ELEMENTS LEAD TO SPURIOUS MODES AGAIN

Lagrange elements for Maxwell lead to spurious eigenmodes. *Also for MHD* (Maxwell+advection).



$\mathbf{u}_0 = 0$ Hodge Laplacian

$\mathbf{u}_2 = (2 \cos(2x) \sin(2y), 2 \sin(2x) \cos(2y))$ potential

✕: from nodal elements ○: Nédélec (FEEC)

LIMITATIONS OF EIGENVALUE ANALYSIS

Convergence is necessary.

Is it sufficient?

Classical numerical analysis

Convergence of eigenvalues

$$\lambda_h \longrightarrow \lambda.$$

Suppose we compute

$$\lambda_h = 10^{-12}.$$

Is it really a dynamo?

Fundamental difficulty

- ▶ The exact λ is never known.
- ▶ $h, \Delta t \not\rightarrow 0$ in practice.
- ▶ Eigenvalues (even pseudospectra) capture only part of the dynamics.

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Beyond eigenvalue convergence

Long-time dynamics (**trajectory**).

Topology (**invariant**).

Stationary states (**equilibrium**).

How much topology
survives
advection?

Beyond classical Hodge theory.

Generalized Hodge theory...

TOPOLOGY OF STATIONARY STATES

Stationary states are natural candidates for long-time asymptotics.

$$\cancel{\partial_t} u + L_\beta u + \eta \Delta_{\text{HL}} u = 0.$$

Classical Hodge theory

Without advection,

$$\dim \ker(\Delta_{\text{HL}}) = \dim \mathcal{H}^k = b_k.$$

The main structures and results of Hodge theory are preserved by FEED.

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Generalized Hodge theory (Arnold, 1982)

$$\dim \ker(L_\beta + \eta \Delta_{\text{HL}}) \geq b_k.$$

Moreover, when the diffusion is sufficiently large.

$$\dim \ker(L_\beta + \eta \Delta_{\text{HL}}) = b_k.$$

FEEC DISCRETIZATION

A natural mixed finite element scheme based on the de Rham complex:

$$\begin{aligned}(\sigma_h, \tau_h) - (i_{\beta,h}^k u_h, \tau_h) - \eta(u_h, d^{k-1} \tau_h) &= 0, \quad \forall \tau_h \in V_h^{k-1}, \\(d^{k-1} \sigma_h, v_h) + \eta(d^k u_h, d^k v_h) + (i_{\beta,h}^{k+1} d^k u_h, v_h) &= 0, \quad \forall v_h \in V_h^k.\end{aligned}$$

Schemes for different degrees are connected!

$$\Delta_h^k = d^{k-1} \delta_h^k + \delta_h^{k+1} d^k, \quad L_{\beta,h}^k = d^{k-1} i_{\beta,h}^k + i_{\beta,h}^{k+1} d^k.$$

For the semidiscrete Lie advection–diffusion problem,

$$\partial_t u_h + A_h^k u_h = 0,$$

the discrete operator has the same algebraic structure as the continuous operator:

$$A_h^k = \eta \Delta_h^k + L_{\beta,h}^k.$$

Most importantly,

$$d^k A_h^k = A_h^{k+1} d^k.$$

FEEC preserves the discrete de Rham complex.

Consequently, Arnold's topological argument can be carried over to the discrete setting.

DISCRETE ARNOLD THEOREM

Theorem 2

The closed discrete stationary space satisfies

$$\dim \ker(A_h^k) \geq b_k.$$

$$Z_h^k := \ker(d^k), \quad B_h^k := \text{ran}(d^{k-1}), \quad S_h^k := \ker(A_h^k|_{Z_h^k}).$$

Proof

Step 1. If $\omega_h \in Z_h^k$, then $d\omega_h = 0$, hence

$$A_h^k \omega_h = d(i_{\beta,h} \omega_h + \eta \delta_h \omega_h) \in B_h^k.$$

Therefore

$$A_h^k(Z_h^k) \subset B_h^k.$$

Step 2. By rank–nullity, for $A_h^k : Z_h^k \rightarrow Z_h^k$,

$$\dim S_h^k = \dim Z_h^k - \dim A_h^k(Z_h^k).$$

Step 3. Since $A_h^k(Z_h^k) \subset B_h^k$,

$$\dim \ker(A_h^k) \geq \dim S_h^k \geq \dim Z_h^k - \dim B_h^k = \dim \mathcal{H}_h^k = b_k.$$

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Continuous proof
(V.I. Arnold, E.I. Korkina)

Fredholm index argument

$$\operatorname{ind}(\eta \Delta_{HL} + L_\beta) = \operatorname{ind}(\Delta_{HL}) = 0$$

therefore

$$\dim \ker A = \dim \operatorname{coker} A.$$



Finite-dimensional analogue
Rank-nullity

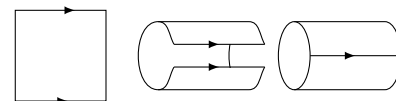
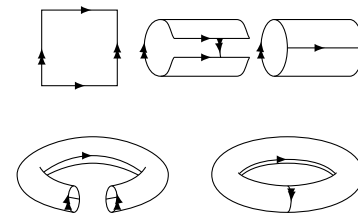
GENERALIZED HARMONIC FORMS

$$\dim \ker(A_h^k) \geq b_k \text{ is sharp.}$$

Torus ($b_1 = 2$)					Cylinder ($b_1 = 1$)				
wind	10^{-1}	1	10	10^2	wind	10^{-1}	1	10	10^2
0	2	2	2	2	0	1	1	1	1
1	2	2	2	2	1	1	1	1	1
2	2	2	2	2	2	1	1	1	1
3	2	2	2	3	3	1	1	1	1
4	2	2	2	3	4	1	1	1	1
5	2	2	2	2	5	1	1	1	1

Number of computed stationary modes ($|\lambda| < 10^{-8}$).

label	velocity field $\mathbf{u}(x)$ in $\mathbb{R}^3$	$\mathbf{u} = \nabla \phi$ on $\mathbb{R}^3$
0	$\mathbf{u} = \mathbf{0}$	yes
1	constant $(1, 1, 1)$	yes
2	$(\cos y + \cos z, \cos z + \sin x, \cos x + \sin y)$	no
3	$(\sin z + \cos y, \sin x + \cos z, \sin y + \cos x)$	no
4	$\nabla \times \mathbf{A}$ with $\mathbf{A} = (\sin y \sin z, \sin z \sin x, \sin x \sin y)$	no
5	Heaviside $(\text{sign}(z - \pi/2), 0, 0)$	no



Extra generalized harmonic forms can appear for general advection.

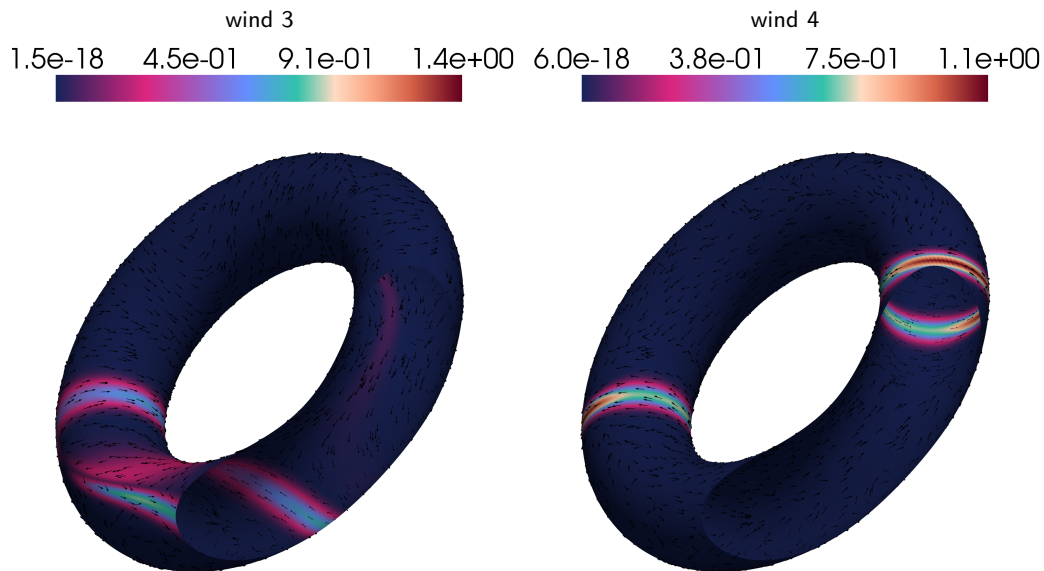


Figure. L^2 orthogonal complement of $\text{span}\{\omega_\theta, \omega_\phi\}$ inside the rank-3 subspace and normalised to unit L^2 ; surface colour is the pointwise field magnitude, with arrows showing the tangent vector field.

POTENTIAL WIND: WHEN $\dim \ker(A^k) = \mathcal{H}^k$ HOLDS

$$\begin{array}{ccccccc}
 \dots & \longrightarrow & \Lambda^{k-1} & \xrightarrow{d} & \Lambda^k & \xrightarrow{d} & \Lambda^{k+1} \xrightarrow{d} \dots \\
 & & \downarrow e^{\theta(x)} & & \downarrow e^{\theta(x)} & & \downarrow e^{\theta(x)} \\
 \dots & \longrightarrow & \Lambda^{k-1} & \xrightarrow{d_\theta} & \Lambda^k & \xrightarrow{d_\theta} & \Lambda^{k+1} \xrightarrow{d_\theta} \dots
 \end{array}$$

$$\beta^b = d\theta \quad \implies \quad d_\theta := e^{-\theta} d e^\theta, \quad \delta_\theta := e^\theta \delta e^{-\theta}.$$

Diagram commutes:

$$e^\theta d(e^{-\theta} w) = dw - d\theta \wedge w.$$

$$d\delta_\theta + \delta_\theta d = \Delta_{\text{HL}} + \mathbf{L}_{\nabla\theta}.$$

Advection–diffusion becomes a weighted Hodge Laplacian.

Consequence:

Hodge theory holds with weighted inner product

$$\beta^b = d\theta \quad \implies \quad \text{self-adjoint after weighting} \quad \implies \quad \dim \ker(A^k) = \dim \mathcal{H}_\theta^k = b_k.$$



Edward Witten

*Supersymmetry and Morse
theory*

Witten deformation
Witten complex
Witten Laplacian

A LOCAL WITTEN TRANSFORM

General advection

$$\beta^b \neq d\theta \quad \implies \quad \text{no global Witten transform.}$$

Idea

Approximate the velocity field by

$$\beta \longmapsto \beta_h \in \mathcal{P}_0.$$

Since $\beta_h = \nabla\psi_h$ locally, apply the Witten transform simplex-by-simplex.

Local exponential-fitting operator

$$\tilde{J}_h^k = \eta E_h^{k+1} dW_h^k,$$

$$W_h^k u = \Pi_h^k(e^{\psi_h/\eta} u), \quad E_h^k = (W_h^k|_{V_h^k})^{-1}.$$

Local Witten transform $\implies$ an approximation of transported exterior derivative $\eta d + j_\beta$.

j_β : adjoint of contraction i_β

WHY THIS FLUX?

FEEC

$$(\sigma_h, \tau_h) - (i_\beta u_h, \tau_h) - \eta(u_h, d\tau_h) = 0,$$

$$(\partial_t u_h, v_h) + (d\sigma_h, v_h)$$

$$+ \eta(du_h, dv_h) + (i_\beta du_h, v_h) = 0.$$

Flux

$$(\sigma_h, \tau_h) - (u_h, \eta d + j_\beta \tau_h) = 0,$$

$$(\partial_t u_h, v_h) + (d\sigma_h, v_h)$$

$$+ (du_h, \eta d + j_\beta v_h) = 0.$$

$$\tilde{J}_h^{k+1} \tilde{J}_h^k = 0.$$

Local Witten

$$(\sigma_h, \tau_h) - (u_h, \tilde{J}_h^k \tau_h) = 0,$$

$$(\partial_t u_h, v_h) + (d\sigma_h, v_h)$$

$$+ (du_h, \tilde{J}_h^k v_h) = 0.$$

For $k = 0$,

A_h is a singular irreducible M-matrix.

Hence, by Perron–Frobenius,

$$\lambda_0 = 0 \text{ simple}, \quad \Re \lambda < 0, \quad e^{-tA_h} \rightarrow P_{\ker A_h}.$$

0-forms

n -forms

$$u_h(t) \longrightarrow \text{constant}.$$

$$\omega_h(t) \longrightarrow \omega_\infty,$$

same total mass.

Correct long-time dynamics at the endpoint degrees.

CLOSED 1-FORMS: COHOMOLOGY DETERMINES THE LIMIT

Exact forms decay

Using

$$A_h^1 d = dA_h^0,$$

we have

$$e^{-tA_h^1} du_h = d\left(e^{-tA_h^0} u_h\right).$$

Since 0-forms converge to constants,

$$du_h(t) \longrightarrow 0.$$

(Hence $\ker(A_h^1) \cap B_h^1 = \{0\}$.)

Dimension argument

Arnold's theorem gives

$$\dim \ker(S_h^1) \geq b_1 = \dim \mathcal{H}_h^1.$$

Since

$$\ker(A_h^1) \cap B_h^1 = \{0\},$$

the natural map

$$\ker(S_h^1) \longrightarrow \mathcal{H}_h^1$$

is an isomorphism.

(Recall: $S_h := A_h|_{Z_h}$)

Every cohomology class contains a unique stationary representative.

Since the cohomology class is preserved by the evolution $(A_h^k(Z_h^k) \subset B_h^k)$, every closed initial 1-form converges to [that](#) representative.

FROM ENDPOINT DEGREES TO ANTIDYNAMO

Analysis for 0- and n -forms propagates to other degrees through the commuting property

$$d^k A_h^k = A_h^{k+1} d^k.$$

1-forms

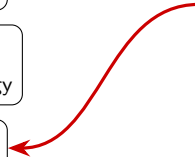
0-forms
converge to constants



exact part

harmonic part
determined by cohomology

coexact part



2-forms
stationary equilibrium

$$n = 2$$

$$1 = n - 1$$

$\Rightarrow$

Finite element antidynamo theorem

$$n = 3$$

$$1 \neq n - 1$$

$\Rightarrow$

Gap at the magnetic-field degree

Only a start...

Towards Computational Topological Hydrodynamics

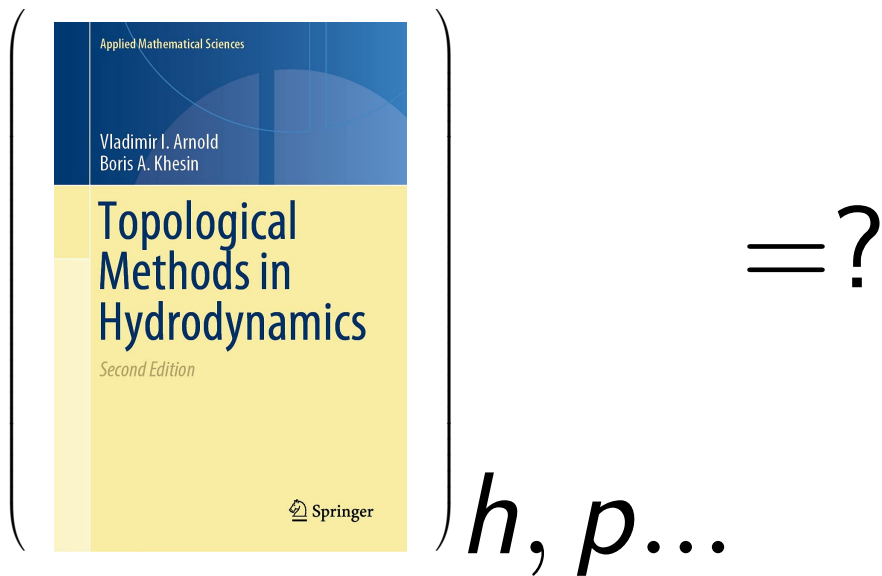
FEEC, Long term computation of fluids/plasma

Topology (knots, braids, cohomology) , **geometry** (Witten complex, generalized Hodge theory) ,
PDE analysis (Onsager conjecture) , **fluid mechanics** (dynamo, turbulence) , **dynamical systems**

+ **numerics**

Long-term computation: finite-dim vs infinite-dim

	Finite-dim Hamiltonian systems	Fluids / plasmas (infinite-dim)
Volume-preserving	phase space	infinite dim. Lie group
Structures	symplectic / contact forms	topology (helicity), geometry (energy, enstrophy)
How many invariants preserve?	as many Casimirs as possible	has a limit (Onsager conjecture, singularities)
Stability	KAM (Feng, Shang 1980s)	dynamo (instability)
Decay/Growth	'rigid'	decay (relaxation), growth (dynamo)
Discretization	geometric integrators	geometric integrators + de Rham complex



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- ▶ *Stable finite element methods preserving $\nabla \cdot \mathbf{B} = 0$ exactly for MHD models*, K. Hu, Y. Ma, J. Xu; *Numerische Mathematik*, 135(2), 371-396 (2017). [divergence-free preservation](#)
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Collaborators: Relaxation



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Collaborators: Dynamo



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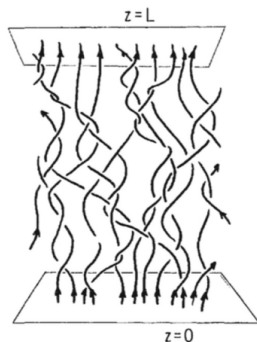
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- *FEEC for advection-diffusion eigenvalue*, D. Boffi, K. Hu, Y. Liang, S. Zampini, U. Zerbini; in preparation
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APPENDIX: NONTRIVIAL COHOMOLOGY & GENERALIZED HELICITY



Additional difficulty (torus topology):

nontrivial cohomology $\Rightarrow$ harmonic field $\mathbf{B}_0 = (0, 0, 1)$

$$\nabla \times \mathbf{B}_0 = 0, \quad \nabla \cdot \mathbf{B}_0 = 0, \quad \text{satisfies BCs}$$

- ▶ Classical helicity not well-defined
- ▶ Harmonic part $\mathbf{B}_H$ is time-invariant ($\partial_t \mathbf{B} \in \mathcal{R}(\text{curl})$) but interferes with exact part

Generalized helicity (gauge-dependent but conserved):

$$\mathbf{B} = \mathbf{B}_R + \mathbf{B}_H, \quad \mathbf{B}_R = \text{curl } \mathbf{A}$$

$$\tilde{H} := \int \mathbf{A} \cdot (\mathbf{B}_R + 2\mathbf{B}_H) = \int \mathbf{A} \cdot (\mathbf{B} + \mathbf{B}_H) \implies \frac{d}{dt} \tilde{H} = 0$$

Generalized Arnold inequality:

$$|\tilde{H}| \leq C(\|\mathbf{B}_R\|^2 + 4\|\mathbf{B}_2 + 2\mathbf{B}_H\|^2)$$

Our algorithm: exactly preserves the discrete analogue of the above.

$\mathbf{B}$ - VS. $\mathbf{A}$ -BASED FORMULATIONS

Maxwell eigenproblem

$$\nabla \times \nabla \times \mathbf{B} = \lambda \mathbf{B}, \quad \nabla \cdot \mathbf{B} = 0.$$

Equivalent formulations

$$\mathbf{B} \iff \mathbf{j} = \text{curl } \mathbf{B} \iff \mathbf{B} = \text{curl } \mathbf{A}.$$

$$\nabla \times \nabla \times \mathbf{j} = \lambda \mathbf{j}, \quad \nabla \cdot \mathbf{j} = 0.$$

$$\nabla \times \nabla \times \mathbf{A} = \lambda \mathbf{A}, \quad \nabla \cdot \mathbf{A} = 0.$$

Compatible finite element discretizations are also equivalent
(de Rham complex / half Hodge–Laplacian).

MHD / advection–diffusion

Lie advection destroys the equivalence.

$$-\text{curl}(\mathbf{u} \times \text{curl } \mathbf{B}) + R_m^{-1} \text{curl curl } \mathbf{B} = \lambda \mathbf{B}, \quad \nabla \cdot \mathbf{B} = 0.$$



$$-\mathbf{u} \times (\text{curl } \mathbf{A}) + R_m^{-1} \text{curl curl } \mathbf{A} + \nabla p = \lambda \mathbf{A}, \quad \nabla \cdot \mathbf{A} = 0.$$

Advantage: requiring only $\mathbf{u} \in L^\infty$ (not $\nabla \mathbf{u} \in L^\infty$).

Limitation: Topology (harmonic 2-forms) is lost.

Maxwell: all natural formulations are equivalent.

MHD with potential $\mathbf{A}$: regularity is gained, cohomology is lost.