

TOWARDS FINITE ELEMENT TENSOR CALCULUS (FETC)

FROM *Computational Topological Hydrodynamics* TO *discrete geometric structures*

Kaibo Hu

University of Oxford

FoCM, Vienna, 14 July, 2026



Collaborators: Douglas Arnold (Minnesota), Andreas Čap (Vienna), Snorre Christiansen (Oslo), Patrick Farrell (Oxford), Jay Gopalakrishnan (Portland), Mingdong He (Oxford), Ting Lin (Peking), Michael Neunteufel (Vienna), Jan Nordbotten (Bergen), Joachim Schöberl (Vienna), Ganghui Zhang (HK PolyU)
+ more

Aknowledgement

Royal Society University Research Fellowship (URF\R1\221398)

ERC starting grant (101164551): **GeoFEM** (Geometric Finite Element Methods)

Steve Smale

The Mathematics of Time

Essays on
Dynamical Systems,
Economic Processes,
and Related Topics



SPRINGER-VERLAG
New York Heidelberg Berlin

Topology, Dynamics, Computation

The Mathematics & Computation of Time

In the Spirit of Smale

“A basic idea behind the design of numerical schemes is that they can *preserve the properties of the original problems* as much as possible...
Different representations for the same physical law can lead to different computational techniques in solving the same problem, which can produce *different numerical results...*”

– Feng Kang

Long-term dynamics of Hamiltonian systems:

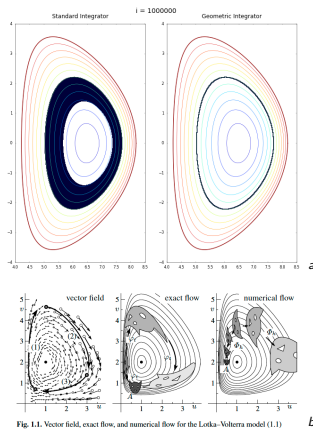
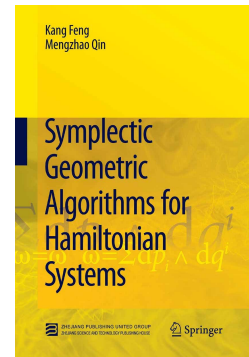
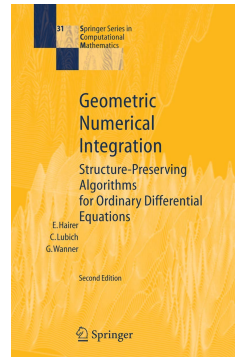


Fig. 1.1. Vector field, exact flow, and numerical flow for the Lotka-Volterra model (1.1)



(Finite dimensional) Hamiltonian systems

$$\omega = \sum_{i=1}^n dq_i \wedge dp_i \quad (\text{canonical symplectic form})$$

A numerical method should be **symplectic**, i.e.

$$\Phi_h^* \omega = \omega$$

geometric numerical integration: 1980s – now
 global problems and global numerical analysis in
 differential equations

^aMax-Planck Institute for Plasma Physics, Numerical Methods in Plasma Physics

^bGeometric Numerical Integration (Hairer, Lubich, Wanner)

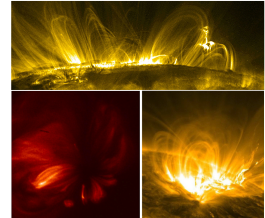
FLUID AND PLASMA SYSTEMS: LONG-TERM EVOLUTION

Plasma equilibria: solar corona

heating of solar corona, self-organization etc.

Parker hypothesis (Still Open)

For “almost any initial data”, *the magnetic field develops tangential discontinuities (current sheet) during the relaxation to static equilibrium.*



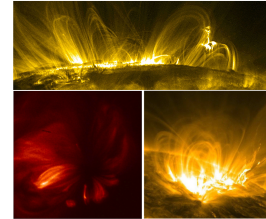
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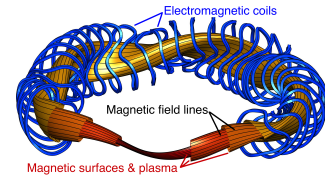
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Plasma equilibria: stellarator, Tokamak

Force-free state $\nabla p = \mathbf{j} \times \mathbf{B}$, $\nabla \cdot \mathbf{B} = 0$.



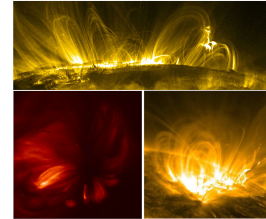
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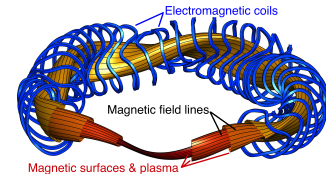
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Instability, dynamo: generation of magnetic fields in stars

Kinematic dynamo: given $\mathbf{u}$, exponential growth of $\mathbf{B}$

$$\partial_t \mathbf{B} - \nabla \times (\mathbf{u} \times \mathbf{B}) - R_m^{-1} \nabla \times \nabla \times \mathbf{B} = 0$$

Simplified magnetohydrodynamics
(MHD):

$$\mathbf{B}_t - \nabla \times (\mathbf{u} \times \mathbf{B}) = 0,$$

$$\mathbf{j} = \nabla \times \mathbf{B},$$

$$\mathbf{u} = \tau \mathbf{j} \times \mathbf{B}.$$

Energy decay

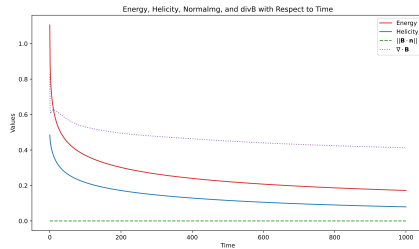
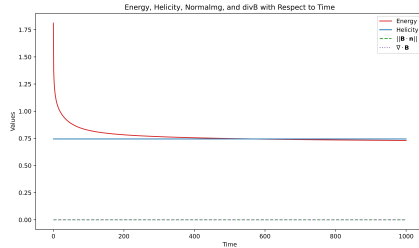
$$\frac{1}{2} \frac{d}{dt} \|\mathbf{B}\|^2 = -\tau \|\mathbf{B} \times \mathbf{j}\|^2.$$

Helicity conservation

$$\frac{d}{dt} \mathcal{H}_m = 0, \quad \mathcal{H}_m := \int \mathbf{A} \cdot \mathbf{B} \, dx, \text{ with } \mathbf{B} = \nabla \times \mathbf{A}.$$

Simplified magnetohydrodynamics (MHD):

$$\begin{aligned} \mathbf{B}_t - \nabla \times (\mathbf{u} \times \mathbf{B}) &= 0, \\ \mathbf{j} &= \nabla \times \mathbf{B}, \\ \mathbf{u} &= \tau \mathbf{j} \times \mathbf{B}. \end{aligned}$$

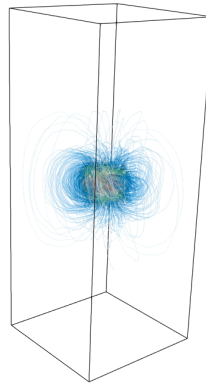


Energy decay

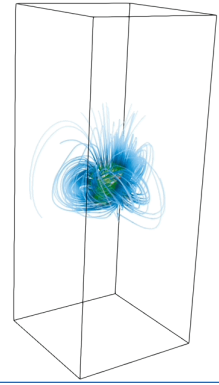
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Helicity-preserving scheme



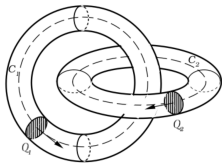
Plain FEM

HELICITY: FINE STRUCTURES

Magnetic helicity: for any potential $\mathbf{A}$ with $\nabla \times \mathbf{A} = \mathbf{B}$

$$\mathcal{H}_m := \int_{\Omega} \mathbf{A} \cdot \mathbf{B} \, dx$$

Topological interpretation: helicity measures linking and knottedness of $\mathbf{B}$



$$\mathcal{H}_{\xi} = 2\ell(C_1, C_2)Q_1 \cdot Q_2$$

ℓ : Gauss linking number, Q_i : flux

Helicity = average asymptotic linking number (Arnold)

Helmholtz–Kelvin; Woltjer; Moffatt

Linked flux tubes $\Rightarrow$ nonzero helicity

Helicity is a key topological invariant in fluid and MHD flows.

A TOPOLOGICAL MECHANISM

Arnold inequality (V.I. Arnold 1974): helicity provides lower bound for energy

$$\left| \int \mathbf{A} \cdot \mathbf{B} dx \right| \leq C \int |\mathbf{B}|^2 dx$$

Proof. Cauchy-Schwarz $|\int \mathbf{A} \cdot \mathbf{B} dx| \leq \|\mathbf{A}\|_{L^2} \|\mathbf{B}\|_{L^2}$ + Poincaré inequality $\|\mathbf{A}\|_{L^2} \leq C \|\nabla \times \mathbf{A}\|_{L^2}$.



Vladimir I. Arnold

Differential form point of view: $\mathbf{A}$: 1-form, $\mathbf{B}$: 2-form

$$\int \mathbf{A} \wedge \mathbf{B}$$

$$\leq C$$

$$\int \mathbf{B} \wedge * \mathbf{B}$$

Helicity, Topology

Energy, Geometry

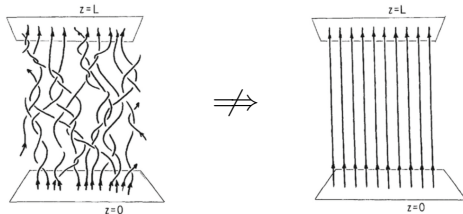


Fig: Pontin, Hornig, Living Rev. Sol. Phys. 2020.

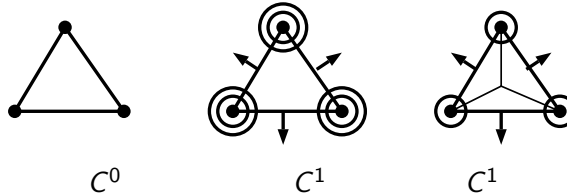
Topology (helicity) constrains energy.

knots are topological barriers that prevent energy from dissipation

Topological mechanism lost of algorithm does not preserve helicity.

How to discretize and preserve helicity?

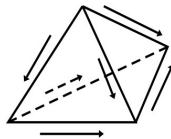
Finite elements: from **scalar fields** (50s, 60s)



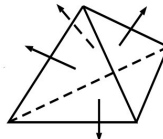
Degrees of Freedom (DoFs): *discrete patterns*, where quantities are discretized
shape functions: extend DoFs to piecewise functions inside

Finite elements = p.w. polynomials with continuity determined by DoFs

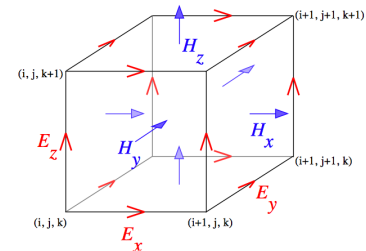
to **vector fields** (70s-80s)



Nédélec

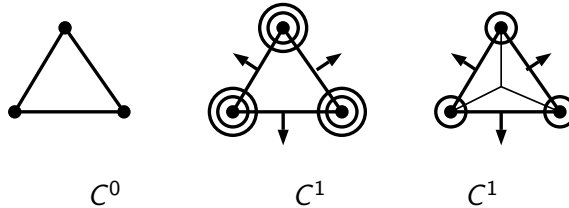


Raviart-Thomas



How to discretize and preserve helicity?

Finite elements: from **scalar fields** (50s, 60s)

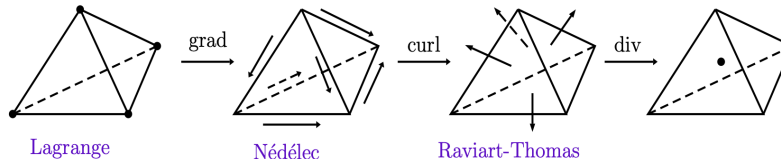


Degrees of Freedom (DoFs): *discrete patterns*, where quantities are discretized
shape functions: extend DoFs to piecewise functions inside

Finite elements = p.w. polynomials with continuity determined by DoFs

to **differential forms** (Bossavit, Hiptmair 80s – 90s; Arnold, Falk, Winther 2000s...)

k-forms on k-cells



$\text{grad } \mathbf{Lag} \subset \mathbf{Ned}$, $\text{curl } \mathbf{Ned} \subset \mathbf{RT}$, $\text{div } \mathbf{RT} \subset \mathbf{DG}$, and exactness / cohomology

Finite Element Exterior Calculus (FEEC)

Whitney forms:

$$W_{i_0 \dots i_k} = k! \sum_{r=0}^k (-1)^r \lambda_{i_r} d\lambda_{i_0} \wedge \dots \wedge \widehat{d\lambda_{i_r}} \wedge \dots \wedge d\lambda_{i_k}.$$

Finite element periodic table: any dimension, any k -forms, any polynomial degrees

Periodic Table of the Finite Elements

The chart displays a periodic table of finite elements, organized by dimension (0D, 1D, 2D, 3D) and polynomial degree (0, 1, 2, 3). The elements are categorized into four families: P (Polynomial), R (Raviart-Thomas), Q (Quadratic), and S (Spline). Each entry includes a diagram of the element shape, its name, and a brief description of its properties.

Legend:

- P_k : Polynomial of degree k
- R_k : Raviart-Thomas of degree k
- Q_k : Quadratic of degree k
- S_k : Spline of degree k

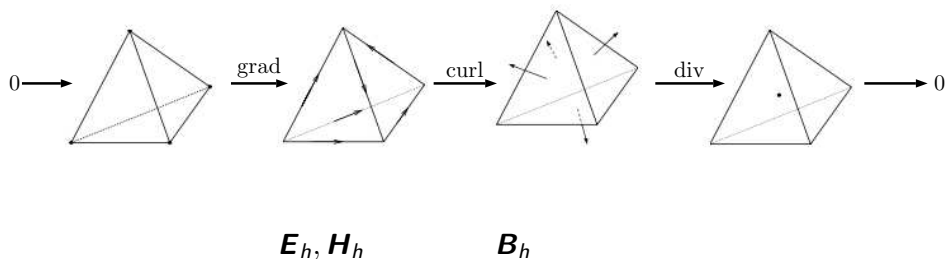
Finite elements:

- P_0 : Constant
- P_1 : Linear
- P_2 : Quadratic
- P_3 : Cubic
- R_0 : Raviart-Thomas of degree 0
- R_1 : Raviart-Thomas of degree 1
- R_2 : Raviart-Thomas of degree 2
- R_3 : Raviart-Thomas of degree 3
- Q_0 : Quadratic of degree 0
- Q_1 : Quadratic of degree 1
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- Q_3 : Quadratic of degree 3
- S_0 : Spline of degree 0
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- S_2 : Spline of degree 2
- S_3 : Spline of degree 3

References:

- 1. J. N. Kutz, "Finite Element Exterior Calculus: Foundations for Harmonic Maps and Meshes," Cambridge University Press, 2013.
- 2. J. N. Kutz, "Finite Element Exterior Calculus: Foundations for Harmonic Maps and Meshes," Cambridge University Press, 2013.
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- 10. J. N. Kutz, "Finite Element Exterior Calculus: Foundations for Harmonic Maps and Meshes," Cambridge University Press, 2013.

PRESERVING HELICITY: VARIABLES IN DISCRETE DE RHAM COMPLEX



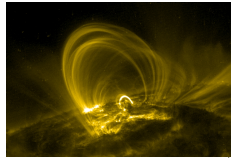
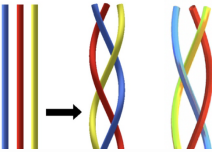
For MHD (previous example)

- ▶ Faraday's law $\partial_t \mathbf{B}_h + \nabla \times \mathbf{E}_h = 0$ holds exactly $\implies \frac{d}{dt}(\nabla \cdot \mathbf{B}_h) = 0$.
- ▶ Introducing projection $\mathbf{H}_h = Q_{L^2} \mathbf{B}_h \implies (\mathbf{u}_h \times \mathbf{H}_h, Q_{L^2} \mathbf{B}_h) = 0$.

+ Any quadratic-invariant-preserving time integrator (implicit midpoint, etc.)

WHY NOT LAGRANGE MULTIPLIERS?

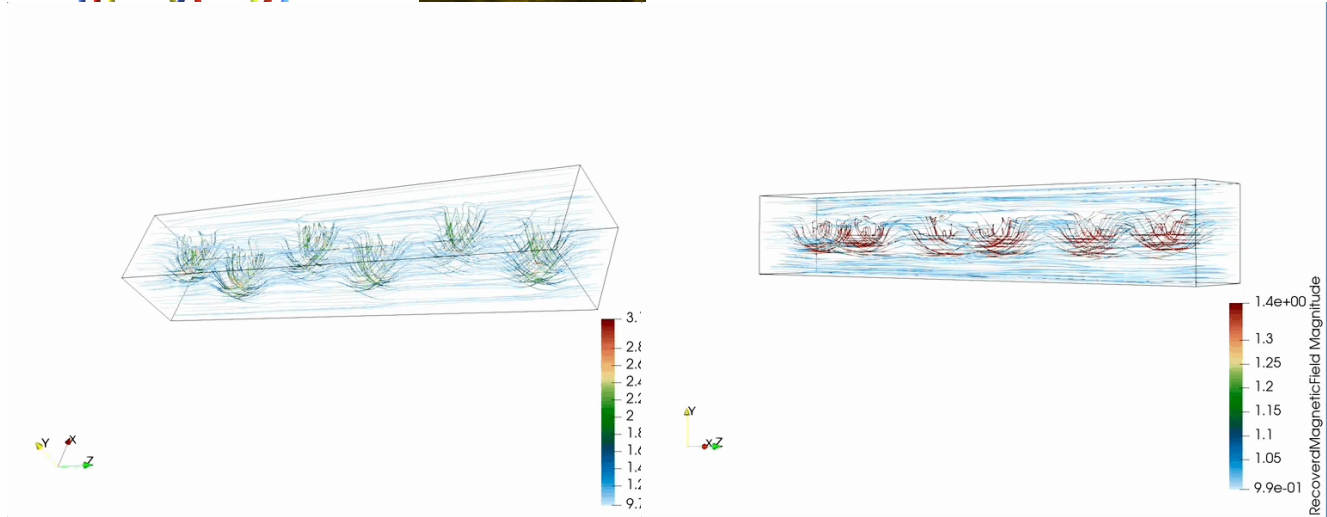
“E3”



nontrivial topology with zero global helicity.

- ▶ magnetic braids,
- ▶ coronal loops.

Pontin–Wilmot-Smith–Hornig–Yeates



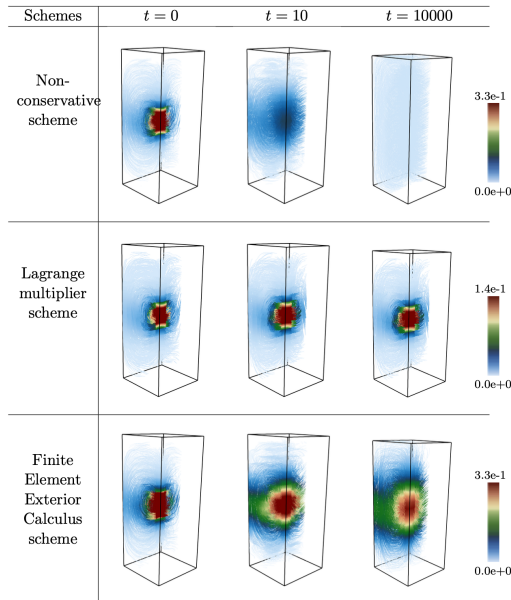
Left: helicity-preserving algorithm Right: preserving integrated $\int \mathbf{A} \cdot \mathbf{B} \, dx$ by Lagrange multiplier.

Broader question: evolve *table of knots* (no theoretical guarantee for numerics yet).

SUBTLETIES IN INFINITE DIMENSIONAL DYNAMICS: RECONNECTION

Unlike finite dimensional systems, invariant-preservation has a limit for PDEs!

True physics always has diffusion $\Rightarrow$ reconnection.



► Selective decay:

helicity decays much slower than energy

► Taylor's conjecture:

global helicity conserved,
local helicity not

**Different topology constraints
 $\Rightarrow$ different equilibria.**

Which one corresponds to real physics?

(more stories, leading to discussions on
turbulence...)

Magnetic knots (Hopf fibration): evolution under different topological constraints

CAN COMPUTATION ANSWER THE DYNAMO QUESTION?

... It is **still unknown** whether this field (ABC flow) is a fast kinematic dynamo, e.g., whether an exponentially growing mode of B survives as $R_m \rightarrow \infty$.

...

Numerically, the kinematic fast dynamo problem is the **first eigenvalue problem** for matrices of the order of many million, even for reasonable Reynolds numbers (of the order of hundreds). The physically meaningful magnetic Reynolds numbers R_m are of order of magnitude 10^8 . **The corresponding matrices are (and will remain) beyond the reach of any computer.**

— Topological Methods in Hydrodynamics, Arnold & Khesin (1998)

$$\nabla \times (\mathbf{u} \times \mathbf{B}) - R_m^{-1} \nabla \times \nabla \times \mathbf{B} = \lambda \mathbf{B}.$$

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“*Eigenvalues* might be meaningful in theory, but they **cannot always be trusted** on a computer. ... And to study nonnormality, a fruitful technique is to look **beyond eigenvalues to pseudospectra**, the sets in the complex plane bounded by level curves of the norm of the resolvent. **pseudospectra.**”

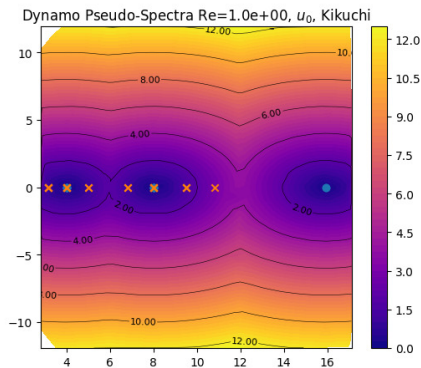
— Spectra and pseudospectra, Trefethen & Embree (2005)

COMPUTING EIGENVALUES AND PSEUDOSPECTRA

ε -pseudospectra

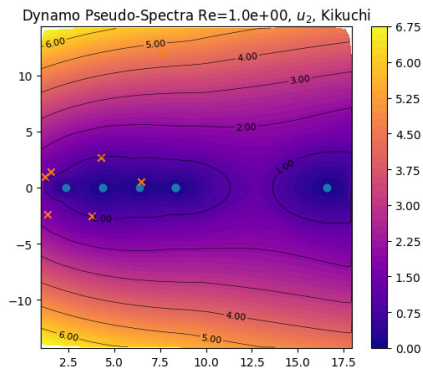
$$\sigma_\varepsilon(A) = \left\{ z \in \mathbb{C} : \|(z - A)^{-1}\| > \varepsilon^{-1} \right\} = \left\{ z \in \mathbb{C} : \lambda_{\min}(z - A) < \varepsilon \right\}.$$

$\lambda_{\min}$: smallest singular value.



$\mathbf{u} = 0$ (Hodge Laplacian)

×: nodal elements



$\mathbf{u} \neq 0$ (advection-diffusion)

○: Nédélec (FEEC)

Finite elements not fitting in de Rham complexes lead to spurious solutions.

Rigorous analysis on eigenvalues and pseudospectra: Boffi, KH, Liang, Zerbini 2026, in preparation

The story continues...

Towards Computational Topological Hydrodynamics
FEEC, Long term computation of fluids/plasma

Topology (knots, braids, cohomology) , geometry (Witten complex, generalized Hodge theory) ,
PDE analysis (singularity, Onsager conjecture) , fluid mechanics (dynamo, turbulence) ,
dynamical systems (long term asymptotics)

+ numerics (global numerical analysis)

17:30–18:30 semi-plenary talk



Kaibo Hu (University of Oxford)

› Towards Computational Topological Hydrodynamics

FoCM PDE Section Talk



Slides

RELIABLE NUMERICAL COMPUTATION

Computation is used for **gravitational wave templates** , **fusion devices** , **quantum computing** , and many other scientific applications.

How confident are we in what we compute?

Key idea: **Differential complexes** encode the *compatibility relations* underlying PDE models.

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$$\dots \longrightarrow V^{k-1} \xrightarrow{d^{k-1}} V^k \xrightarrow{d^k} V^{k+1} \longrightarrow \dots, \quad d^k \circ d^{k-1} = 0.$$

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Fundamental example: the de Rham complex

$$0 \longrightarrow C^\infty(\Omega) \xrightarrow{\text{grad}} C^\infty(\Omega; \mathbb{R}^3) \xrightarrow{\text{curl}} C^\infty(\Omega; \mathbb{R}^3) \xrightarrow{\text{div}} C^\infty(\Omega) \longrightarrow 0.$$

$$\text{curl grad} = 0, \quad \text{div curl} = 0.$$

and generalizations to **differential forms**.

Finite Element Exterior Calculus (FEEC) provides a canonical discretization of the de Rham complex.

BEYOND DIFFERENTIAL FORMS

Many problems are tensorial.

Continuum mechanics

- ▶ stress
- ▶ strain
- ▶ dislocation density
- ▶ disclination density

$$\sigma_{ij} = \sigma_{ji}$$

Differential geometry

- ▶ metric
- ▶ curvature
- ▶ torsion

$$g_{ij} = g_{ji}$$

$$R_{ab;cd} = -R_{ba;cd} = -R_{ab;dc}, \quad R_{ab;cd} = R_{cd;ab}, \\ R_{ab;cd} + R_{bc;ad} + R_{ca;bd} = 0.$$

Differential forms are only **one class of tensors** (fully skew symmetric)

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Differential forms are only **one class of tensors** (fully skew symmetric)

Can we discretize general tensor fields while preserving

algebraic symmetries + **differential structures?**

Beyond de Rham

When skew-symmetry is no longer enough,
what differential structure is available?

Symmetric tensor complexes and the BGG construction

AN EXAMPLE BEYOND DE RHAM: THE ELASTICITY COMPLEX

$$\mathbb{V} := \mathbb{R}^3, \quad \mathbb{S} := \mathbb{R}_{\text{sym}}^{3 \times 3}.$$

$$\text{RM} \hookrightarrow C^\infty(\Omega; \mathbb{V}) \xrightarrow{\text{sym grad}} C^\infty(\Omega; \mathbb{S}) \xrightarrow{\text{inc}} C^\infty(\Omega; \mathbb{S}) \xrightarrow{\text{div}} C^\infty(\Omega; \mathbb{V}) \longrightarrow 0$$

rigid motions

displacement

metric / strain

curvature / stress

force

$$\text{inc} \circ \text{sym grad} = 0$$

Saint-Venant compatibility

$$\text{div} \circ \text{inc} = 0$$

Bianchi identity

Fundamental complex for symmetric tensor fields.

Discovered independently from several perspectives:

- ▶ Kröner (defect theory)
incompatibility tensor models dislocations
- ▶ Calabi: linearized Riemannian geometry
metrics $\rightarrow$ curvature
- ▶ Olver: hyperforms, n -complexes
calculus of variation

The elasticity complex plays the same structural role in continuum mechanics as the de Rham complex does for differential forms.

ELASTICITY COMPLEX IS BUILT UP FROM DE RHAM

Inspired by the Bernstein–Gelfand–Gelfand (BGG) construction

B-G-G (1975), Eastwood (2000), Čap–Slovák–Souček (2001), Arnold–Falk–Winther (2006), Arnold–Hu (2021), Čap–Hu (2024).

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathbb{R}^3 & \xrightarrow{\text{grad}} & \mathbb{R}^{3 \times 3} & \xrightarrow{\text{curl}} & \mathbb{R}^{3 \times 3} & \xrightarrow{\text{div}} & \mathbb{R}^3 & \longrightarrow & 0 \\
 & & & \nearrow S^0 & & \nearrow S^1 & & \nearrow S^2 & & & \\
 0 & \longrightarrow & \mathbb{R}^3 & \xrightarrow{\text{grad}} & \mathbb{R}^{3 \times 3} & \xrightarrow{\text{curl}} & \mathbb{R}^{3 \times 3} & \xrightarrow{\text{div}} & \mathbb{R}^3 & \longrightarrow & 0
 \end{array}$$

Two copies of the de Rham complex
connected by algebraic maps.

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B-G-G (1975), Eastwood (2000), Čap–Slovák–Souček (2001), Arnold–Falk–Winther (2006), Arnold–Hu (2021), Čap–Hu (2024).

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathbb{R}^3 & \xrightarrow{\text{grad}} & \mathbb{S} + \cancel{\mathbb{K}} & \xrightarrow{\text{curl}} & \mathbb{R}^{3 \times 3} & \xrightarrow{\text{div}} & \cancel{\mathbb{R}^3} & \longrightarrow & 0 \\
 & & & \nearrow \text{-mskw} & & \nearrow \mathbb{S} & & \nearrow 2 \text{ vskw} & & & \\
 0 & \longrightarrow & \cancel{\mathbb{R}^3} & \xrightarrow{\text{grad}} & \mathbb{R}^{3 \times 3} & \xrightarrow{\text{curl}} & \mathbb{S} + \cancel{\mathbb{K}} & \xrightarrow{\text{div}} & \mathbb{R}^3 & \longrightarrow & 0
 \end{array}$$

Cohomology-preserving elimination

Arnold–Hu, *Complexes from Complexes*, FoCM (2021).

Čap–Hu, *BGG sequences with weak regularity and applications*, FoCM (2024).

ELASTICITY COMPLEX IS BUILT UP FROM DE RHAM

Inspired by the Bernstein–Gelfand–Gelfand (BGG) construction

B-G-G (1975), Eastwood (2000), Čap–Slovák–Souček (2001), Arnold–Falk–Winther (2006), Arnold–Hu (2021), Čap–Hu (2024).

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{R}^3 & \xrightarrow{\text{sym grad}} & \mathbb{S} & \xrightarrow{\text{curl}} & \\ & & & & \searrow & \nearrow & \\ & & & & \mathbb{S} & \xleftarrow{\text{div}} & \mathbb{R}^3 \longrightarrow 0 \end{array}$$

div is labeled T above the arrow from $\mathbb{S}$ to $\mathbb{R}^3$.

Output:

Tensor-valued differential complexes
with the **same cohomology** as the de Rham complex.

Examples

- ▶ Elasticity (Calabi, Kröner...)
- ▶ Hessian
- ▶ Conformal
- ▶ ...

Topology, Algebra $\implies$ Analysis

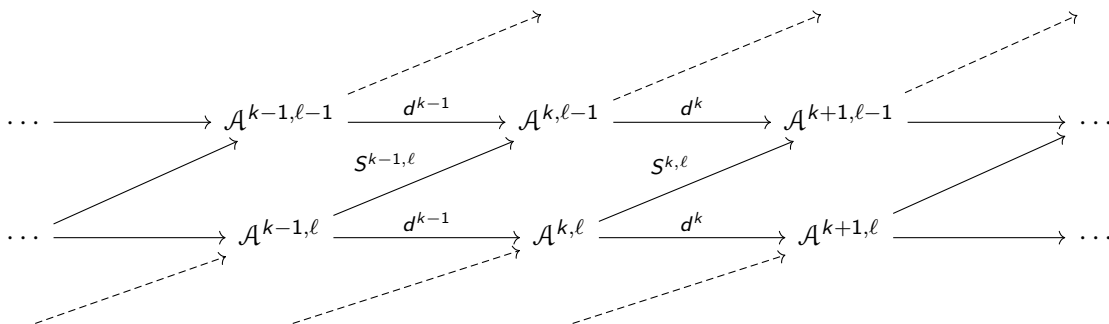
Poincaré–Korn inequalities
Hodge decomposition
Compactness

Arnold–Hu, *Complexes from Complexes*, FoCM (2021).

Čap–Hu, *BGG sequences with weak regularity and applications*, FoCM (2024).

SYSTEMATIC CONSTRUCTION IN A GENERAL SETTING

General principle: eliminate variables while preserving cohomology



Horizontal arrows: differential operators

Diagonal arrows: algebraic maps (symmetrization)

- **Input:** a double complex with horizontal differentials and diagonal symmetry maps.
- **Reduction:** pass to $\ker S$ and $\operatorname{coker} S$ in each degree.
- **Output:** a new complex, often the physically relevant one, with the **same cohomology**.

A CLASS OF EXAMPLES: *form-valued forms*

From differential forms to tensor-valued differential forms

$$\text{Alt}^{k,\ell} := \text{Alt}^k \otimes \text{Alt}^\ell, \quad \ell\text{-form-valued } k\text{-forms (double forms)}.$$

Horizontal arrows are the exterior derivative (grad, curl, div in three dimensions); diagonal arrows encode tensor symmetries.

$$\begin{array}{ccccccc} \dots & \longrightarrow & \text{Alt}^{k-1,\ell} & \xrightarrow{d^{k-1}} & \text{Alt}^{k,\ell} & \xrightarrow{d^k} & \text{Alt}^{k+1,\ell} & \xrightarrow{d^{k+1}} & \text{Alt}^{k+2,\ell} & \longrightarrow & \dots \\ & & \searrow^{S^{k-1,\ell+1}} & & \searrow^{S^{k,\ell+1}} & & \searrow^{S^{k+1,\ell+1}} & & & & \\ \dots & \longrightarrow & \text{Alt}^{k-1,\ell+1} & \xrightarrow{d^{k-1}} & \text{Alt}^{k,\ell+1} & \xrightarrow{d^k} & \text{Alt}^{k+1,\ell+1} & \xrightarrow{d^{k+1}} & \text{Alt}^{k+2,\ell+1} & \longrightarrow & \dots \end{array}$$

metric $g = g_{ij} dx^i \otimes dx^j$ (1, 1)-form

curvature $R = R_{ijkl} dx^i \wedge dx^j \otimes dx^k \wedge dx^l$ (2, 2)-form

Algebraic symmetries encoded in differential complexes

$$\begin{array}{ccccc}
 \text{Alt}^{1,1} & \xrightarrow{d} & \text{Alt}^{2,1} & \xrightarrow{d} & \text{Alt}^{3,1} \\
 & \searrow S^{1,2} & & \searrow S^{2,2} & \\
 \text{Alt}^{1,2} & \xrightarrow{d} & \text{Alt}^{2,2} & \xrightarrow{d} & \text{Alt}^{3,2}
 \end{array}$$

Example: Riemann curvature tensor

The algebraic symmetries

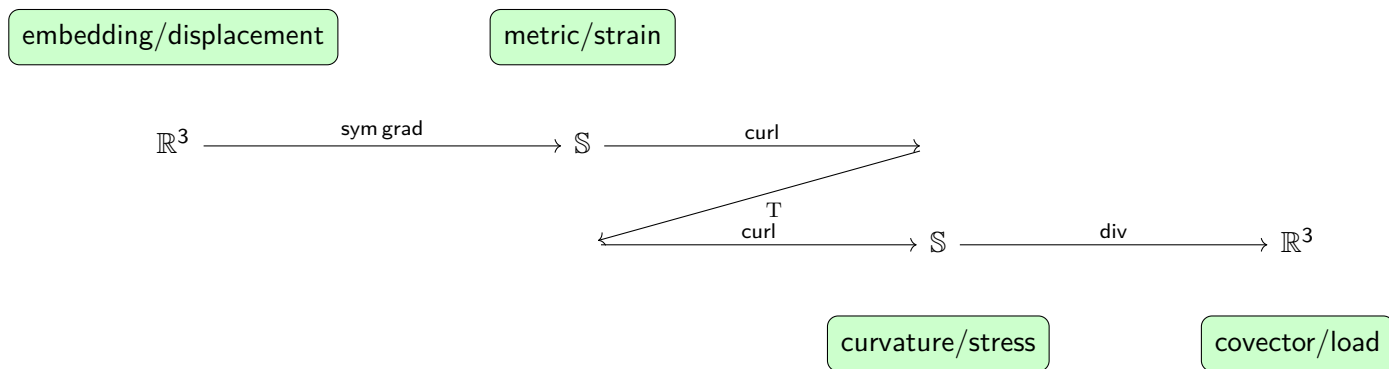
$$R_{ab;cd} = -R_{ba;cd} = -R_{ab;dc}, \quad R_{ab;cd} = R_{cd;ab}, \quad R_{ab;cd} + R_{bc;ad} + R_{ca;bd} = 0,$$

are precisely expressed by

$$\ker(S^{2,2}) \subset \text{Alt}^{2,2}.$$

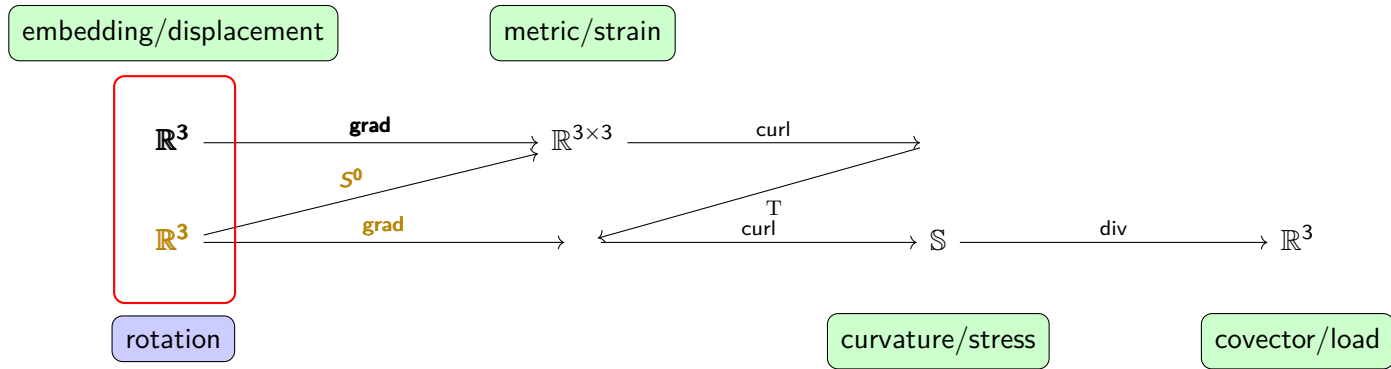
$$\dim \ker(S^{2,2}) = \dim \text{Alt}^{2,2} - \dim \text{Alt}^{3,1} = \begin{cases} 1, & \text{in 2D (Gauss curvature)} \\ 6, & \text{in 3D (Ricci / Einstein)} \\ 20, & \text{in 4D (full Riemann)} \end{cases}$$

THE FULL DIAGRAM ENCODES MORE COMPLETE MODELS

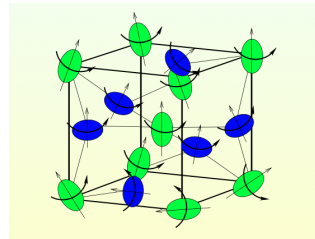


Riemann, Kröner, Cauchy, Hooke

THE FULL DIAGRAM ENCODES MORE COMPLETE MODELS



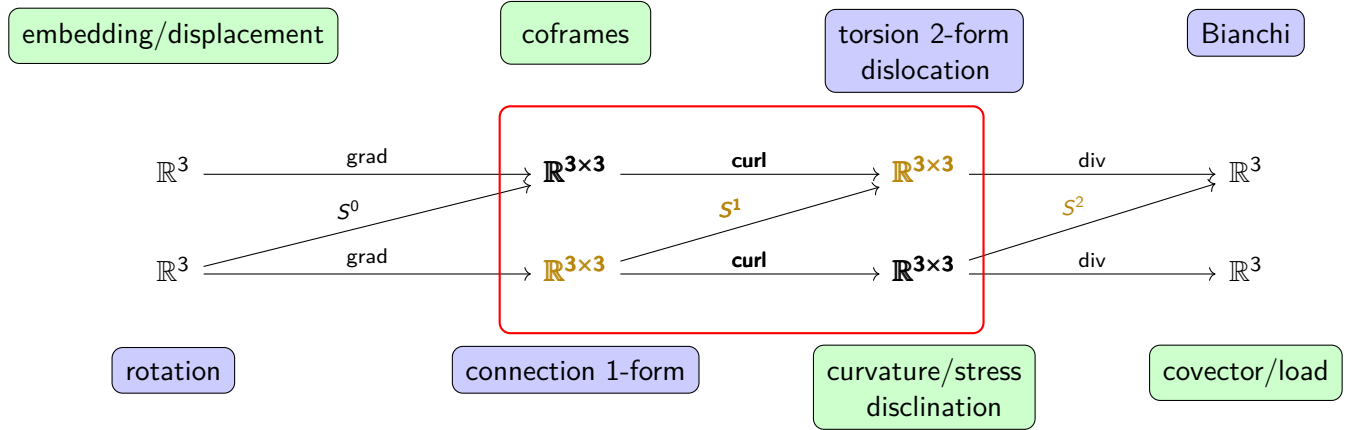
Cosserat brothers



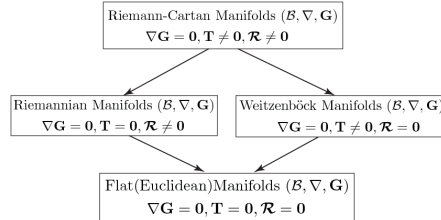
Cosserat continua: microstructures (rotation, stretch etc.)

A. Čap & KH, *BGG sequences with weak regularity and applications*. FoCM (2024).
 Christiansen, KH, & Lin, *Extended Regge complex for linearized Riemann-Cartan geometry and cohomology*. FoCM (2026)

THE FULL DIAGRAM ENCODES MORE COMPLETE MODELS



Élie Cartan



Yavari, Goriely (2012)

BGG construction is thus cohomology-preserving elimination of microstructures

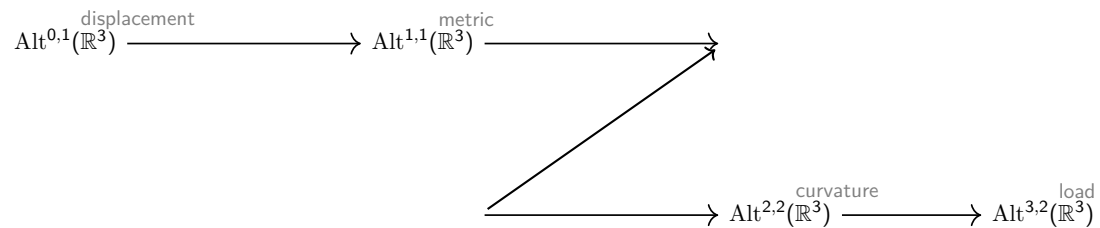
DIFFERENTIAL COMPLEXES AS A UNIFIED MODELLING LANGUAGE

One mathematical language — many continuum models

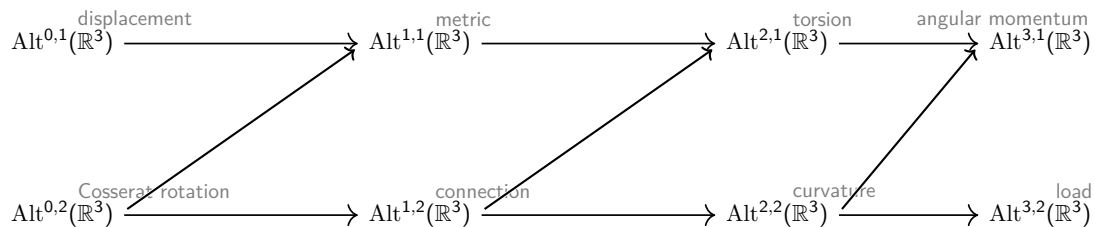
Differential complex	Representative models
de Rham	electromagnetism, diffusion, fluids
BGG (elasticity)	elasticity, plates & shells, general relativity
Twisted (full diagram)	Cosserat / micropolar continua, microstructure, torsion
Higher-order forms	defects, dislocations, disclinations
Čech	contact, fractured media, mixed-dimensional coupling

Geometry, Symmetry $\implies$ Differential complexes $\implies$ Continuum models

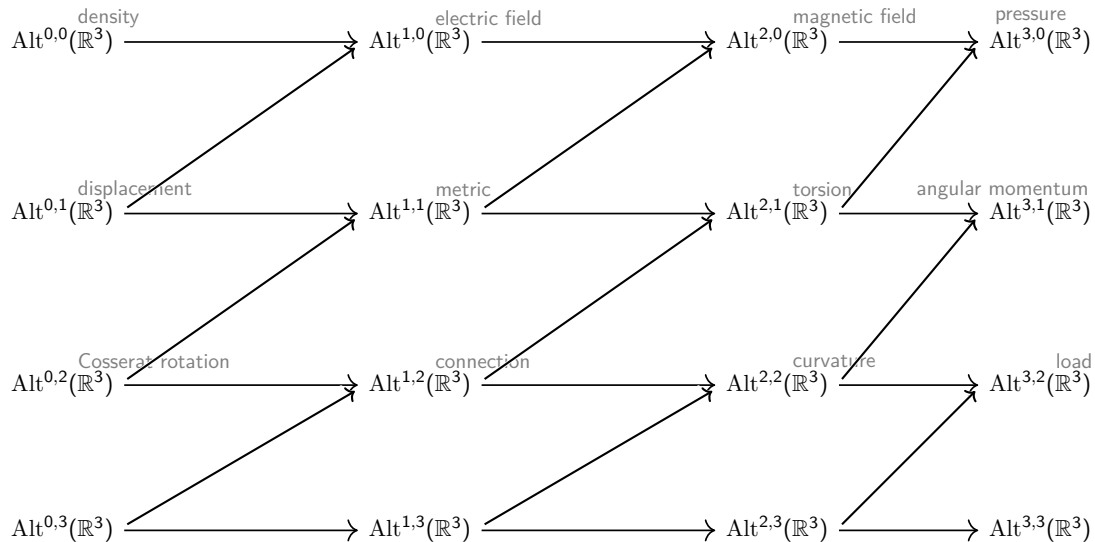
The same geometric structures govern fields, compatibility, energies, topology, and discretizations.



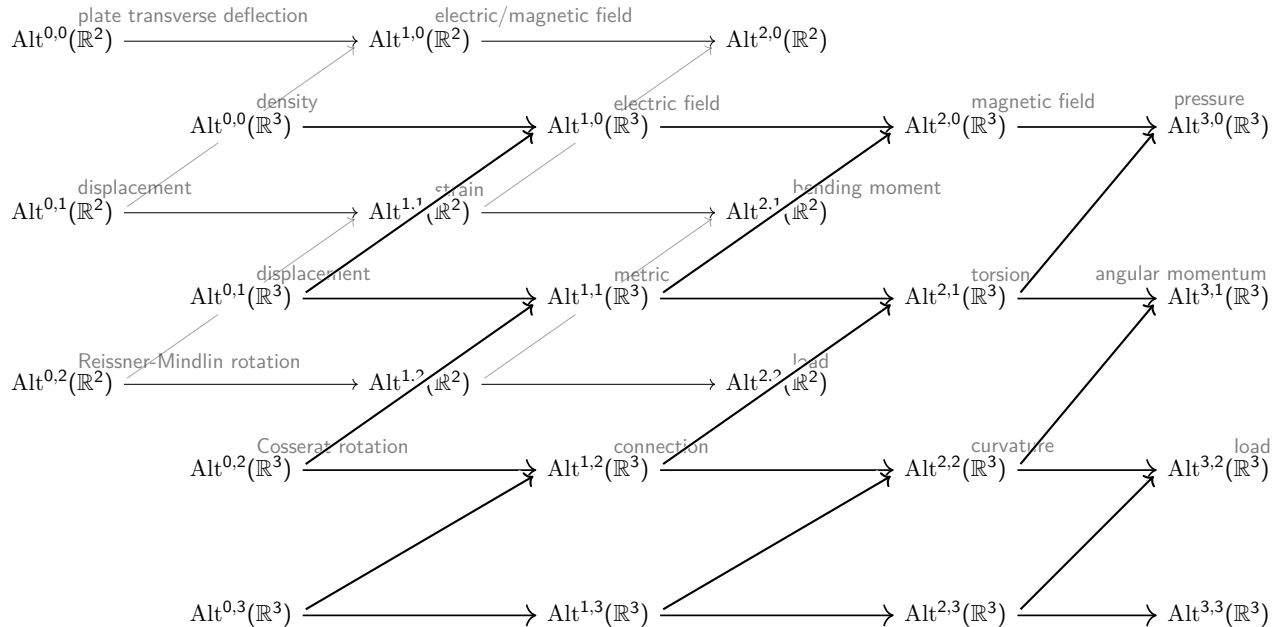
BGG (elasticity, Calabi, Köner) complex



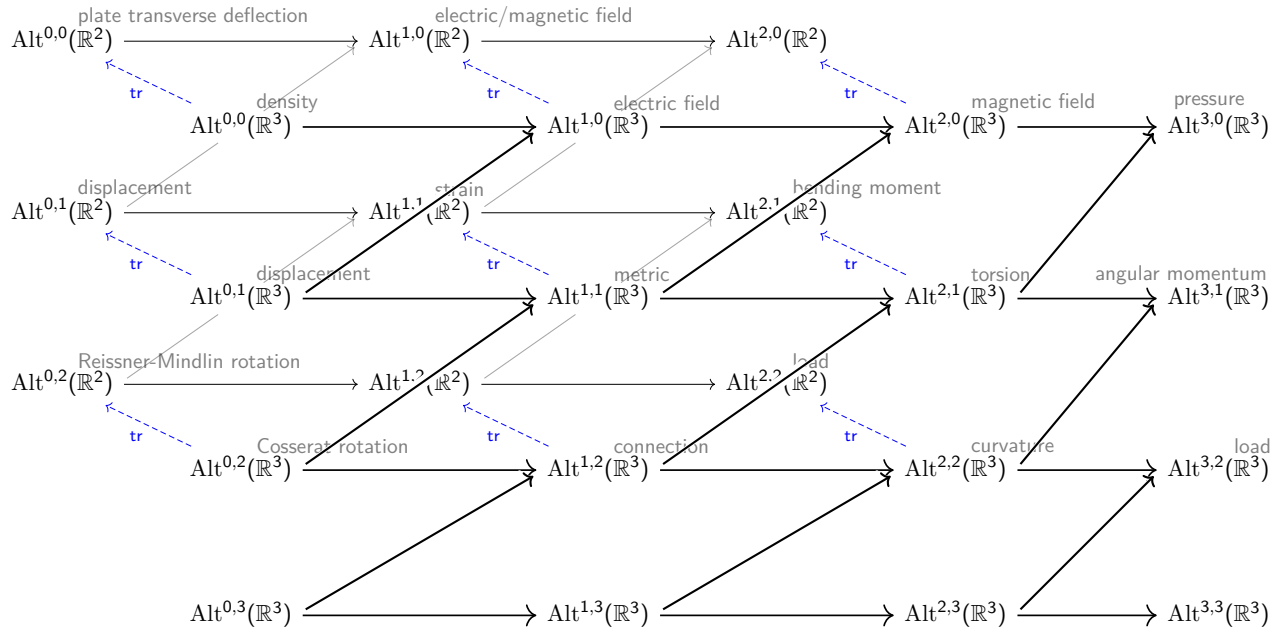
**Twisted de Rham complex: Riemannian-
Cartan geometry, micropolar (Cosserat) media**



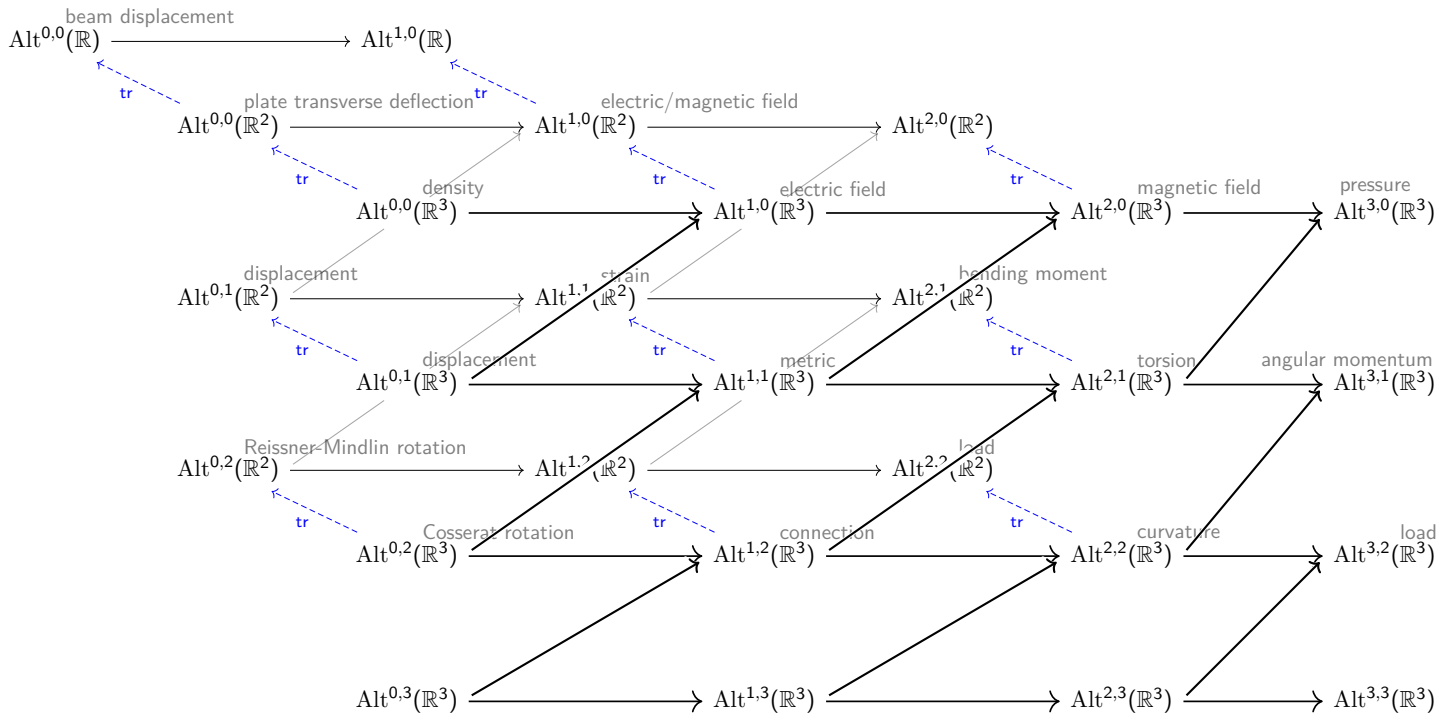
Full BGG diagram (with form-valued forms)



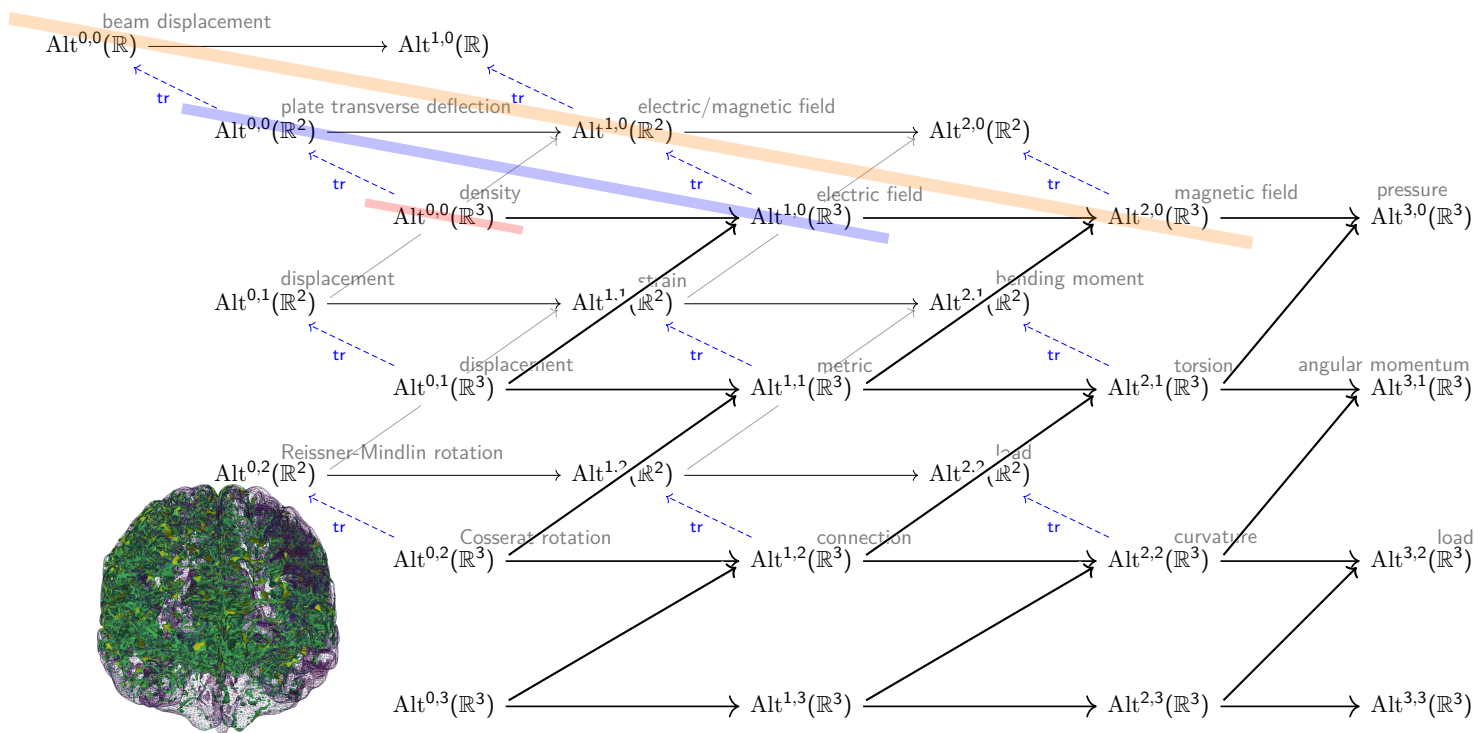
2D: plates



trace, restriction: dimension reduction



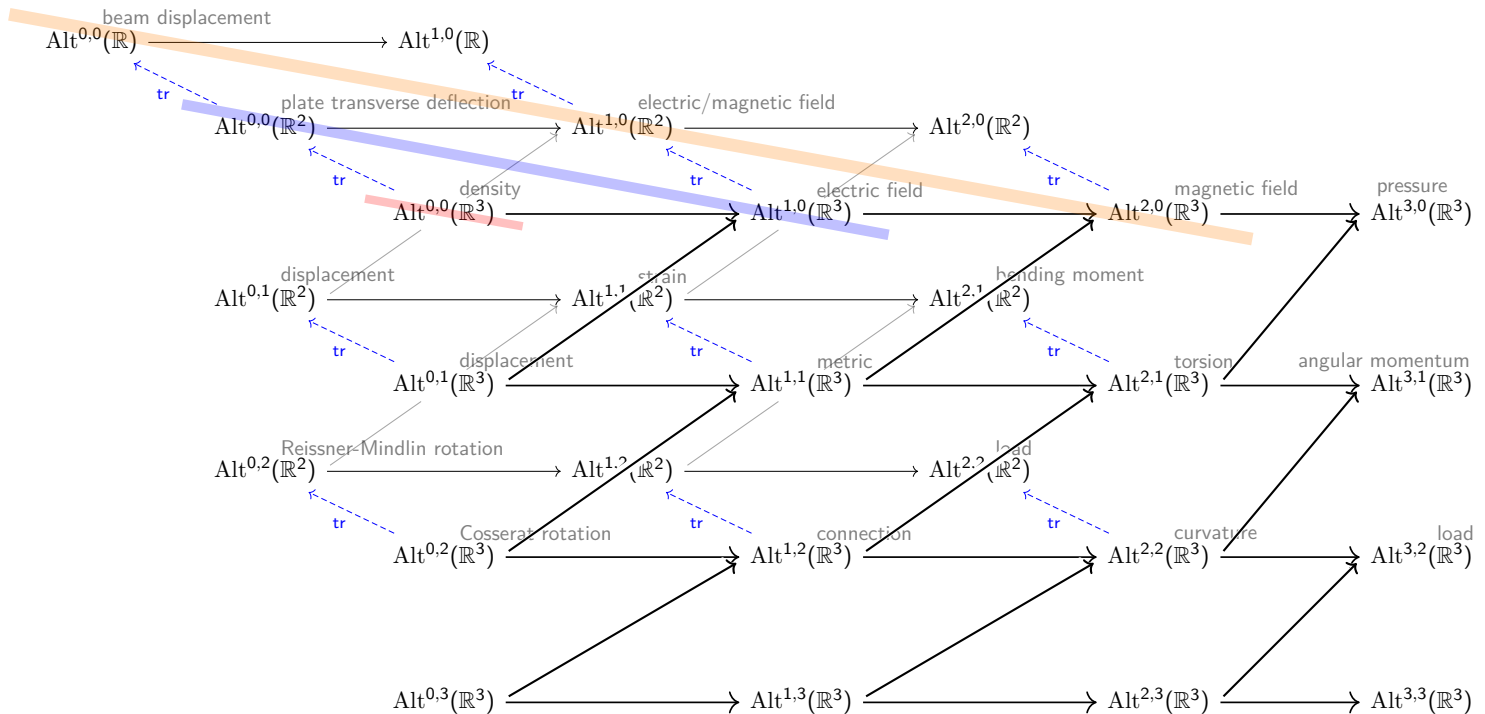
1D: beams



Waterscales: Mathematical and computational foundations for modelling cerebral fluid flow; Marie E. Rognes, EMS.

Boon, Holmen, Nordbotten, Vatne (2025)
Boon & Nordbotten (2023)

**Čech-de Rham double complex:
mixed dimensions, coupling, contact**



BGG+Čech: full mixed dimensional coupling of tensorial models

Ongoing with D.Holmen, J.Nordbotten, J.Vatne

From Continuous to Discrete

De Rham has chains and cochains.

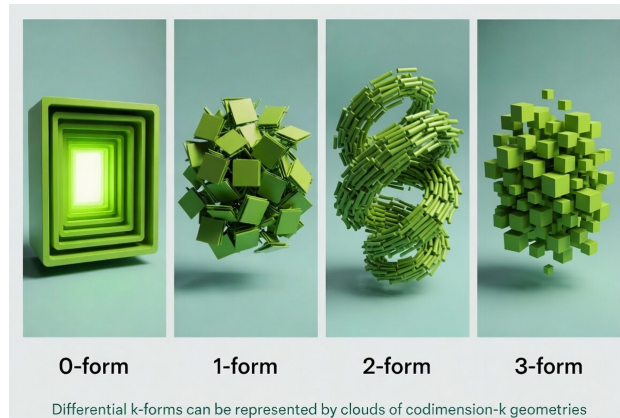
What plays this role for symmetric tensors?

From discrete topology to discrete differential geometry

GENERALIZING FINITE ELEMENTS: BACK TO DE RHAM'S CURRENTS

Key idea: Incorporating **currents** (Dirac deltas, functionals) supported on mesh topological entities

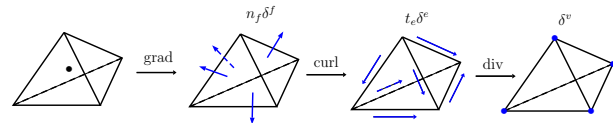
Intuition: codimensional geometry, **k -forms discretized on $(n - k)$ -cells**



Fluid perspective: singular vortex on codim 2



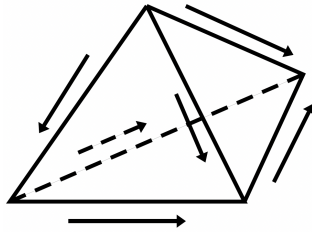
Finite element perspective: currents fit in a complex



Braess, Schöberl 2008, Licht 2017

ELASTICITY COMPLEX, DISCRETE METRIC & CURVATURE

Christiansen 2011: Regge calculus = finite elements

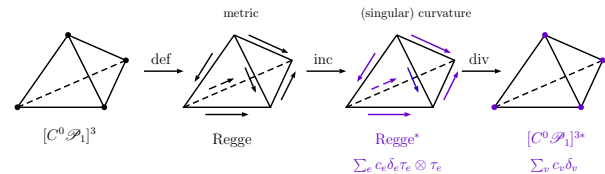
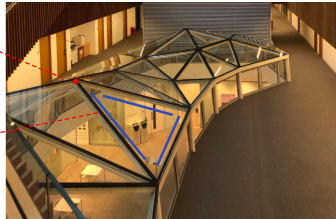


Regge finite element

Finite Element	Discrete Differential Geometry
Metric: p.w. const sym. (0,2)-tensor	Piecewise affine manifold
Degrees of freedom $\int_e g(e, e)$	Edge lengths
t - t continuity	Single-valued edge lengths
distributional curvature (codim 2)	angle deficit
Compatibility ($D^2 = 0$)	Bianchi identities
Exact sequence	Rigidity, 'fundamental thm'

curvature:
Dirac measure on hinges

piecewise constant metric
represented by edge lengths



Cohomology isomorphic to continuous version (Christiansen, KH, Lin 2026, FoCM)

Regge calculus (quantum and numerical GR) $\longleftrightarrow$ Regge finite elements

T.Regge, *General relativity without coordinates*. (1961).

Christiansen, *On the linearization of Regge calculus*. Num. Math. (2011).

GENERAL CONSTRUCTION OF FES FOR FORM-VALUED FORMS

Setting

$\Lambda^{k,\ell}$ (ℓ -form-valued k -forms) with symmetries

Discretized on k -simplices, as in the Whitney case.

Generalized trace (degrees of freedom)

form slot: ι^*

value slot: j^*

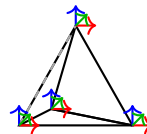
The first k arguments are always tangential.

The value part may involve normals.

Lowest-order examples



(1, 1) Regge



(0, 1)



(1, 2)

covector-valued function

$$\int_e \omega(\mathbf{t})(\mathbf{t}) \, ds$$

$\omega(\cdot)(\mathbf{n}_i)$ at vertices

$$\int_e \omega(\mathbf{t})(\mathbf{t}, \mathbf{n}_i)$$

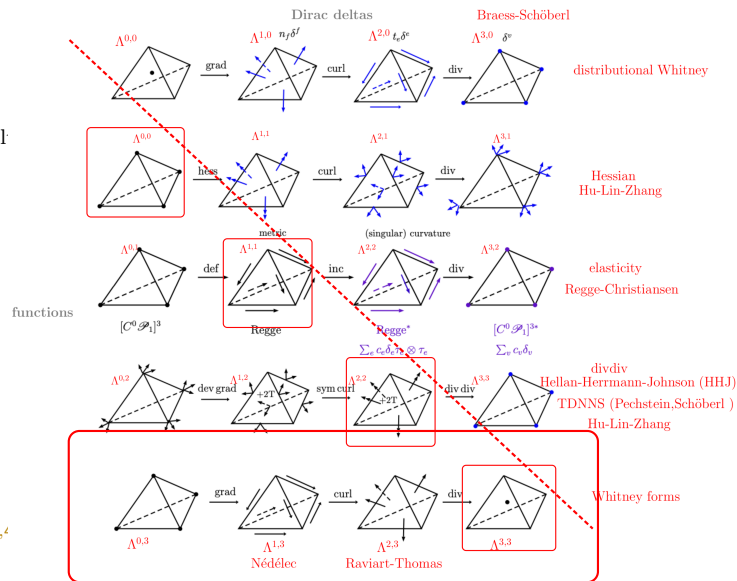
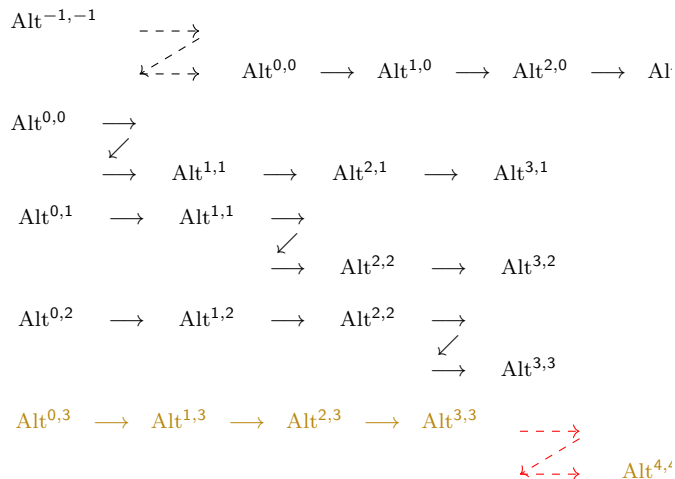
Tensor analogues of Whitney forms: discrete cochains for double forms via generalized traces.

$$\omega(v_1, \dots, v_k)(w_1, \dots, w_\ell) \longmapsto (\iota^* \omega)(v_1, \dots, v_k)(w_1, \dots, w_\ell)$$

with $v_i \in TK$, and $w_j \in TK$ or NK .

A dual picture exists: codimensional version, Dirac measures, k -forms on $(n - k)$ -cells

TOWARDS A FINITE ELEMENT PERIODIC TABLE FOR TENSORS



classical Finite Element Exterior Calculus

Nédélec, Raviart-Thomas, Whitney, Bossavit, Hiptmair, Arnold, Falk, Winther...

Periodic Table of the Finite Elements

The table displays a grid of finite element types, each represented by a small diagram and labeled with its corresponding tensor space. The rows and columns represent different degrees of freedom and tensor ranks. The table is organized into sections for different types of elements (e.g., scalar, vector, tensor) and their corresponding tensor spaces (e.g., $\Lambda^{p,q}$).

TOWARDS A FINITE ELEMENT PERIODIC TABLE FOR TENSORS

$\text{Alt}^{-1,-1}$



$\text{Alt}^{0,0} \longrightarrow \text{Alt}^{1,0} \longrightarrow \text{Alt}^{2,0} \longrightarrow \text{Alt}^{3,0}$

$\text{Alt}^{0,0}$



$\text{Alt}^{1,1}$



$\text{Alt}^{2,1}$



$\text{Alt}^{3,1}$

$\text{Alt}^{0,1}$



$\text{Alt}^{1,1}$



$\text{Alt}^{2,2}$



$\text{Alt}^{3,2}$

$\text{Alt}^{0,2}$



$\text{Alt}^{1,2}$



$\text{Alt}^{2,2}$



$\text{Alt}^{3,3}$

$\text{Alt}^{0,3}$



$\text{Alt}^{1,3}$



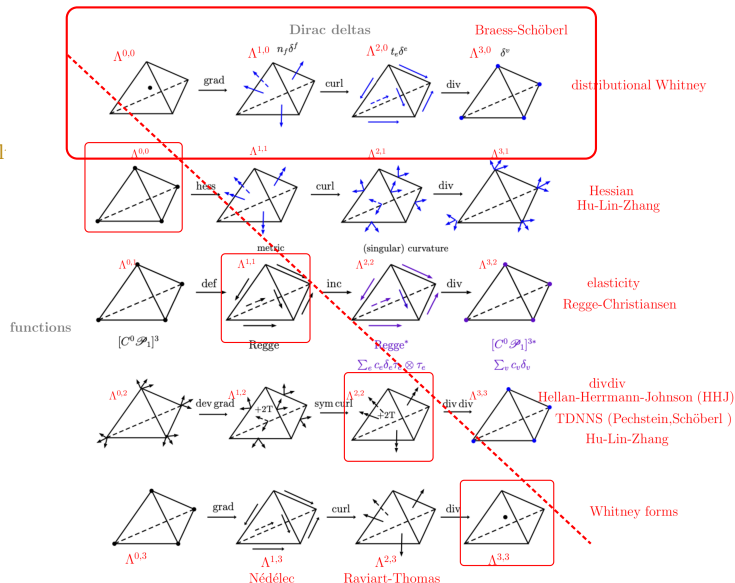
$\text{Alt}^{2,3}$



$\text{Alt}^{3,3}$



$\text{Alt}^{4,4}$



distributional de Rham complex (currents).

Braess, Schöberl 2008: equilibrated residual error estimator

Licht 2017: double complex

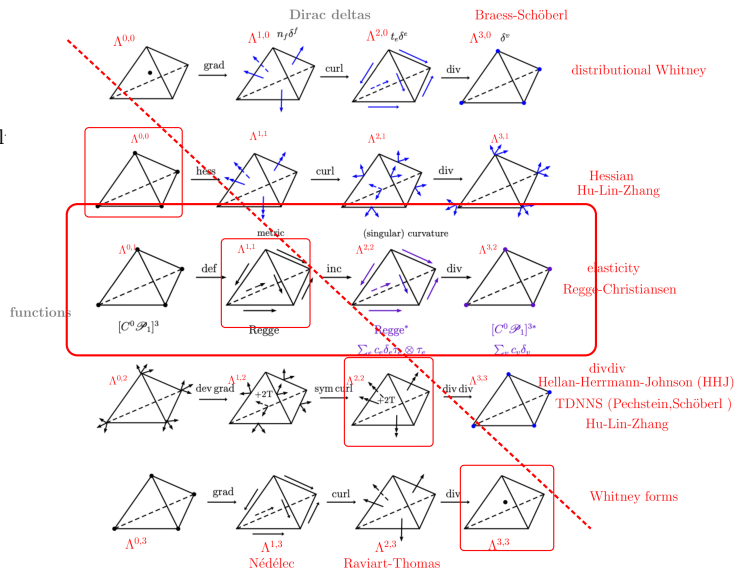
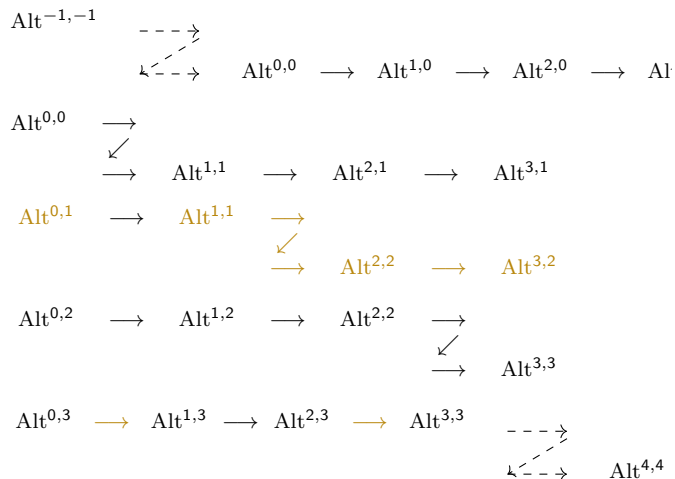


Dietrich Braess



Joachim Schöberl

TOWARDS A FINITE ELEMENT PERIODIC TABLE FOR TENSORS



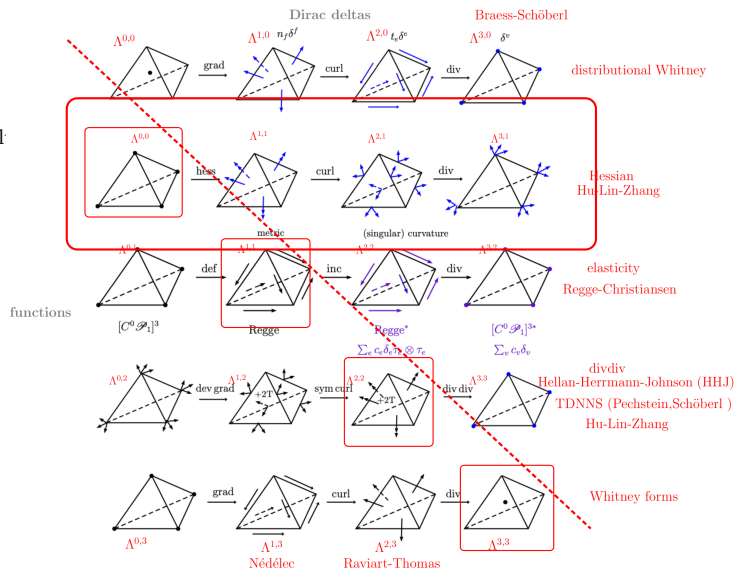
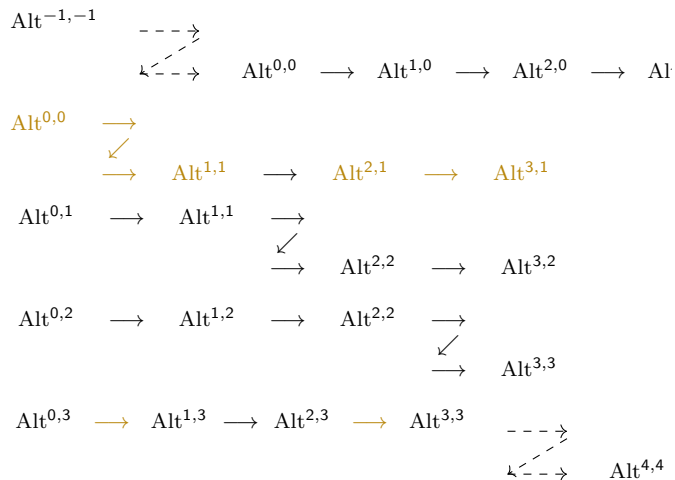
Tullio Regge



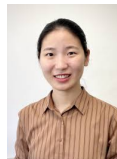
Snorre Christiansen

Christiansen's interpretation of **Regge calculus** as **finite elements**

TOWARDS A FINITE ELEMENT PERIODIC TABLE FOR TENSORS



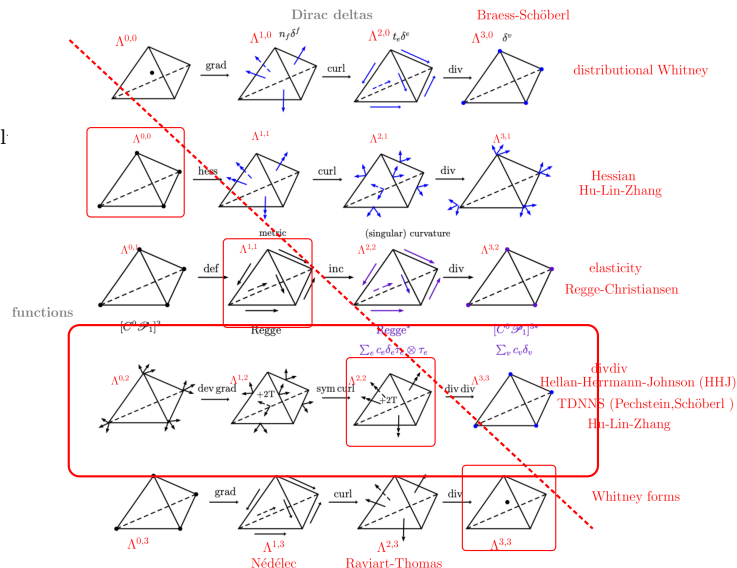
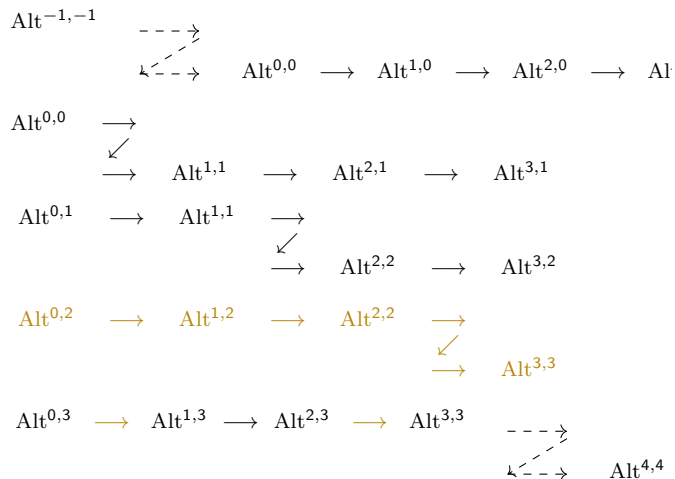
Ting Lin



Qian Zhang

Hessian complex, unified structures identified.

TOWARDS A FINITE ELEMENT PERIODIC TABLE FOR TENSORS



divdiv complex : plate, elasticity (Schöberl, Sinwel 2007), Einstein–Bianchi formulation for relativity.



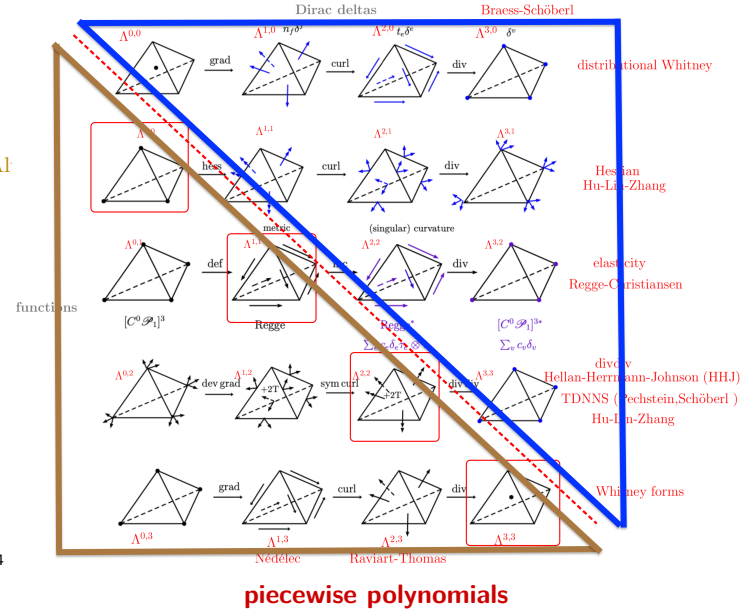
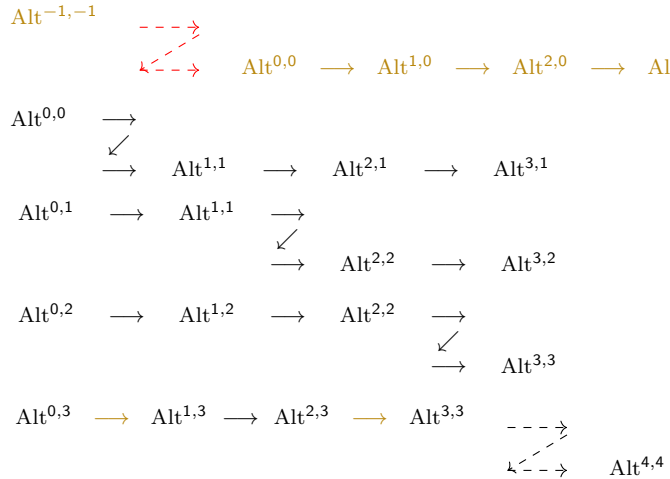
Astrid Pechstein



Joachim Schöberl

TOWARDS A FINITE ELEMENT PERIODIC TABLE FOR TENSORS

distributions, currents, Dirac delta



Patterns, Symmetries, Duality.

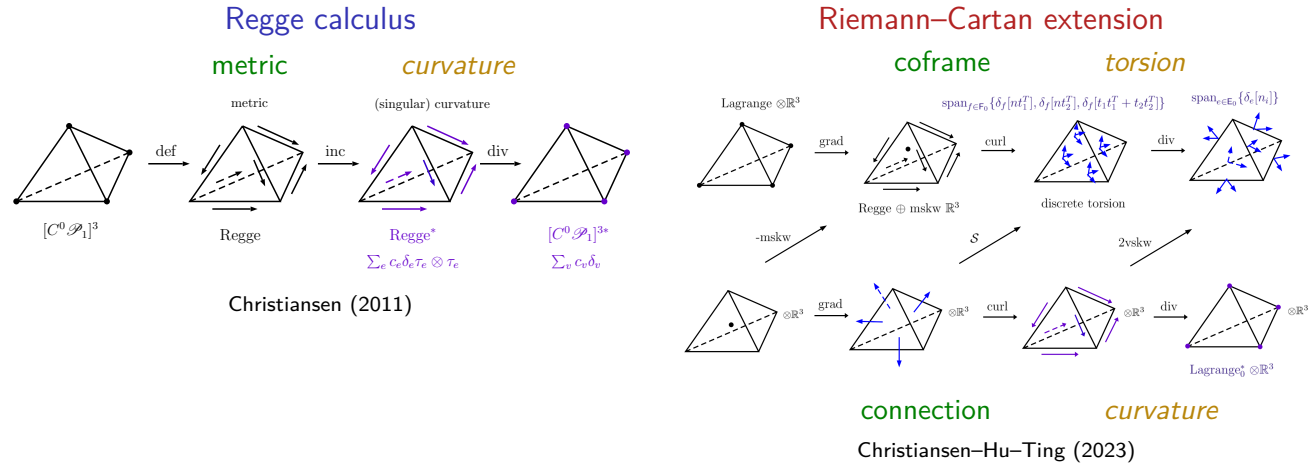
functions (classical finite elements) v.s. Dirac measures (currents). any dimension, any degree.

Classical Finite Element Periodic Table (last row) is the special case of the generalised Table where all spaces are piecewise polynomials.

THE FULL DIAGRAM INCORPORATES A DISCRETE TORSION

BGG diagram encodes both curvature and torsion.

Extending the Regge elasticity complex to the full BGG diagram yields **discrete torsion**.

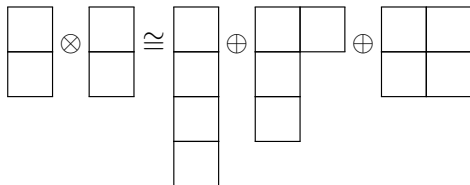


Christiansen, *On the linearization of Regge calculus*. Num. Math. (2011).

Christiansen, KH, & Lin, *Extended Regge complex for linearized Riemann–Cartan geometry and cohomology*. FoCM (2026)

REPRESENTATION THEORY BEHIND TENSOR COMPLEXES

Tensor symmetries are classified by Young tableaux.



Irreducible components

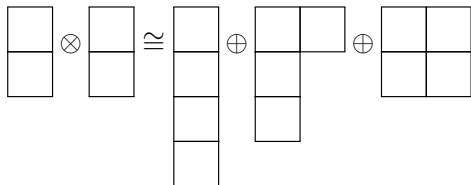
- ▶ 4-forms
- ▶ skew (2, 2)-forms
- ▶ Riemann-type tensors

Canonical projections:

- ▶ Bianchi symmetrization
- ▶ Kulkarni–Nomizu operator

REPRESENTATION THEORY BEHIND TENSOR COMPLEXES

Tensor symmetries are classified by Young tableaux.



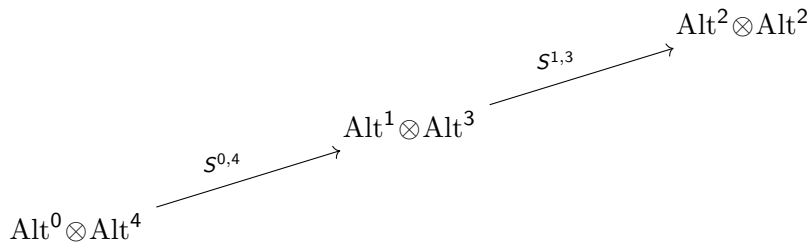
Irreducible components

- ▶ 4-forms
- ▶ skew (2, 2)-forms
- ▶ Riemann-type tensors

Canonical projections:

- ▶ Bianchi symmetrization
- ▶ Kulkarni–Nomizu operator

BGG perspective



BGG, Representation theory $\implies$ continuous and discrete tensor decomposition

APPLICATION: MONGE-AMPÉRE EQUATIONS

Curvature identity:

well defined: moving derivatives to test

$$u_{xx}u_{yy} - u_{xy}^2 = -\partial_x^2(\frac{1}{2}u_y^2) + \partial_x\partial_y(u_xu_y) - \partial_y^2(\frac{1}{2}u_x^2)$$

Gauss curvature, not defined if $u \notin C^1$

- ▶ Ball's existence proof of nonlinear elasticity (compensated compactness)
- ▶ conservation law (divergence structure)

APPLICATION: MONGE-AMPÉRE EQUATIONS

Curvature identity:

$$\operatorname{rot} \operatorname{rot}(\nabla u \otimes \nabla u)$$

$$u_{xx}u_{yy} - u_{xy}^2 = -\partial_x^2(\tfrac{1}{2}u_y^2) + \partial_x\partial_y(u_xu_y) - \partial_y^2(\tfrac{1}{2}u_x^2)$$

$$\det(\operatorname{hess} u)$$

► Peter Olver, *Differential hyperforms I*. (1982)

APPLICATION: MONGE-AMPÈRE EQUATIONS

$$\text{rot rot}(\nabla u \otimes \nabla u)$$

$$u_{xx}u_{yy} - u_{xy}^2 = -\partial_x^2(\frac{1}{2}u_y^2) + \partial_x\partial_y(u_xu_y) - \partial_y^2(\frac{1}{2}u_x^2)$$

$$\det(\text{hess } u)$$

$$\det(\text{hess } u) = f \quad \text{in } \Omega, \quad u = g \quad \text{on } \partial\Omega,$$

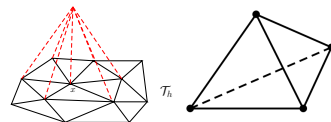
Regge–Lagrange discretization:

$$u_h \in \text{Lagrange}(\mathcal{T}_h), \quad \sigma_h = \nabla u_h \otimes \nabla u_h \in \text{Regge}(\mathcal{T}_h)$$

Weak form:

$$\underbrace{\langle \text{rot rot}(\underbrace{\nabla u_h \otimes \nabla u_h}_{\text{Regge}}), v_h \rangle}_{\text{Lagrange}^*} = (f, v_h), \quad \forall v_h \in \text{Lagrange}_0$$

Lagrange element



Regge element



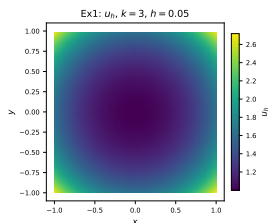
BGG complex



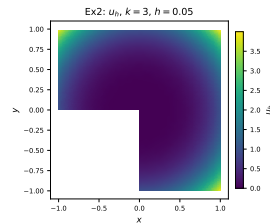
Linear differential complex can solve fully nonlinear equations

CONVERGENCE: INTRINSIC FE FOR MONGE-AMPÈRE

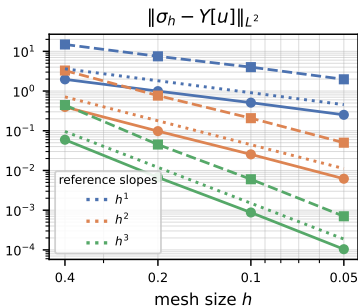
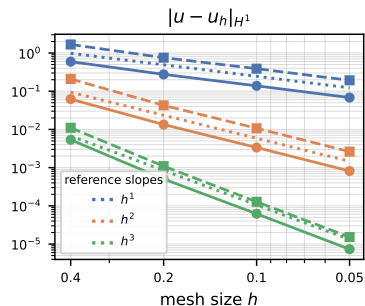
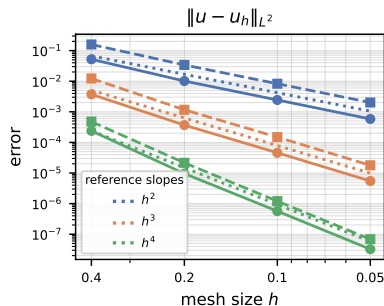
Ex1 (smooth square): $u = e^{(x^2+y^2)/2}$



Ex2 (nonconvex L-domain): $u = (x^2 + y^2)^2$



— $k=1$ — $k=2$ — $k=3$ —●— Ex1 —■— Ex2



Numerical tests using  NGSolve <https://ngsolve.org>

Finite Element Tensor Calculus

FEEC

Differential forms

Discrete topology

FETC

Tensor fields

Discrete geometry

Finite elements are only one realization.

Meshes • Graphs • Networks • Cell complexes • Discrete manifolds • ...

Discretize the geometry,
not individual equations.

Collaborators: Relaxation



Boris
Andrews
Oxford



Patrick
Farrell
Oxford



Mingdong He
Oxford



Ganghui
Zhang
Oxford

- *Topology-preserving discretization for the magneto-frictional equations arising in the Parker conjecture*, M. He, P. E. Farrell, K. Hu, B. D. Andrews; SISC (2025)
- *Global and local helicity-preservation in the finite element discretisation of magnetic relaxation*, P.E. Farrell, M. He, K. Hu, & G. Zhang, arXiv (2026).

Collaborators: Dynamo



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Umberto
Zerbini
Oxford



Jindong Wang
Oxford

- *FEEC for advection-diffusion eigenvalue*, D. Boffi, K. Hu, Y. Liang, S. Zampini, U. Zerbini; in preparation
- *Long-term asymptotics of FEEC for advection-diffusion*, K. Hu, J. Wang, U. Zerbini; in preparation

COLLABORATORS: BGG & DISCRETE GEOMETRY



Douglas
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Christiansen
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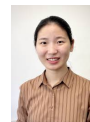
Jay Gopalakr-
ishnan
Portland State



Ting Lin
Peking



Joachim
Schöberl
TU Wien



Qian Zhang
Jilin

- *Complexes from complexes*, D. Arnold, K. Hu; Found. Comput. Math. (2021). [framework & continuous construction](#)
- *BGG sequences with weak regularity and applications*, A. Čap, K. Hu; Found. Comput. Math. (2024). [more general framework, continuum models](#)
- *Distributional Hessian and divdiv complexes on triangulations and cohomology*, K. Hu, T. Lin, Q. Zhang; SIAM J. Appl. Algebra Geom. (2025). [distributional BGG complexes, cohomology](#)
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- *A 2-complex containing Sobolev spaces of matrix fields*, J. Gopalakrishnan, K. Hu, J. Schöberl; arXiv (2025). [matrix-valued complexes](#)
- *Extended Regge complex for linearized Riemann–Cartan geometry and cohomology*, S. H. Christiansen, K. Hu, T. Lin; Found. Comput. Math. (2026). [Cartan geometry & cohomology](#)

A COMMUNITY EFFORT TOWARDS FETC

Related talks at FoCM PDE Workshop:

15:00–15:30



Evan Gawlik (Santa Clara University)

> **Intrinsic and extrinsic perspectives on discrete curvature**

18:00–18:30



Joachim Schöberl (TU Wien)

> **Distributional Finite Elements with Applications for Elasticity, Fluids, and Curvature**

15:30–16:00



Martin Licht (TU Dresden)

> **Estimating Continuous and Discrete Poincaré–Friedrichs–Weber Constants**

14:00–14:30



Jay Gopalakrishnan (Portland State University)

> **Distributional Riemann Curvature Tensor**

14:30–15:00



Long Chen (UC Irvine)

> **Geometric Decompositions of Finite Element Spaces and Complexes**

18:00–18:30



Michael Neunteufel (TU Wien)

> **Finite Element Approximation of the Shape Operator via Discrete Differential Geometry**



ESI Thematic Program (2026, April-June) Differential Complexes: Theory, Discretization, and Applications

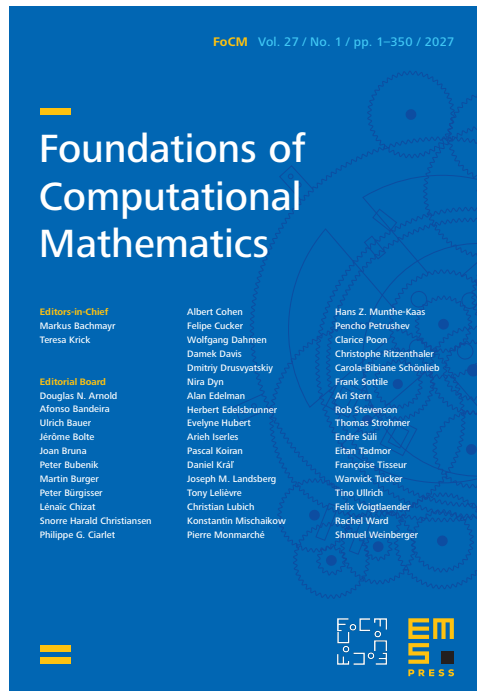
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