Auxiliary Space Theory: Simple Construction of Sophisticated Iterative Methods for Linear Systems

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Abstract

- **Auxiliary space theory**: a unified framework for analyzing various iterative methods for solving linear systems
- Sharp convergence estimates of iterative methods using elementary linear algebra: identities for the error propagation operator and the condition number
- Various applications: subspace correction methods, Hiptmair–Xu preconditioners, saddle point problems, and iterative substructuring methods

References

1. **JP**. Unified analysis of saddle point problems via auxiliary space theory (2025+).
2. **JP** and J. Xu. Auxiliary space theory for the analysis of iterative methods for semidefinite linear systems (2025+).
3. J. Xu and L. Zikatanov. Algebraic multigrid method (Acta Numer. 2017).

Syllabus

1. **Chapter 1: Basic iterative methods**
   - Linear systems and iterative methods
   - Abstract theory of iterative methods
   - Richardson, Jacobi, and Gauss–Seidel methods
   - Steepest descent and conjugate gradient methods
2. **Chapter 2: Auxiliary space theory**
   - Auxiliary space theory
   - Subspace correction methods
   - Hiptmair–Xu preconditioners
3. **Chapter 3: Applications to saddle point problems**
   - Saddle point problems
   - Sharp estimates for Schur complements
   - Augmented Lagrangian method
   - Mixed finite element methods
   - FETI-DP

Chapter 1. Basic Iterative Methods

Table of Contents

- Let V be a finite-dimensional vector space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$.

- Consider the <u>linear system</u>:

$$Au = f, \qquad \text{(Linear)}$$

where $A\colon V \to V$ is a SPD linear operator and $f \in V$.

  - $A = A^t$,
  - $(Av, v) \geq \mu\|v\|^2$ for any $v \in V$, for some $\mu > 0$.

Proposition 1

Given $u \in V$, it is a solution to the linear system (Linear) *if and only if it solves the quadratic optimization problem*

$$\min_{v \in V} \left\{ J(v) := \frac{1}{2}(Av, v) - (f, v) \right\}.$$

A two-point boundary value problem

Consider the 1D Poisson equation:

$$-u'' = f, \quad x \in \Omega := (0,1), \qquad u(0) = 0,\ u'(1) = 0.$$

Variational formulation

- Introduce the function space *(Sobolev space)*:

$$V = \{v \in C(\overline{\Omega}) : v \text{ is piecewise differentiable}, v(0) = 0\}.$$

- Variational formulation: Find $u \in V$ such that

$$a(u,v) = (f,v) \quad \forall v \in V,$$

  where

$$a(u,v) = \int_0^1 u'v'\,dx, \quad u, v \in V.$$

- Consider a uniform partition of $(0,1)$ with grid points

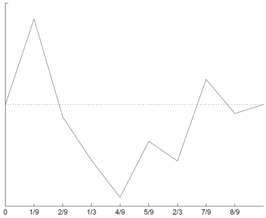
$$x_j = jh, \quad (j = 0, \ldots, N,\ h = 1/N).$$

- Define the linear finite element space:

$$V_h = \{v \in C(\overline{\Omega}) : v|_{(x_{j-1}, x_j)} \text{ is linear},\ 1 \le j \le N,\ v(0) = 0\}.$$

- Nodal basis functions $\{\varphi_i(x)\}_{i=1}^N$ are defined as:

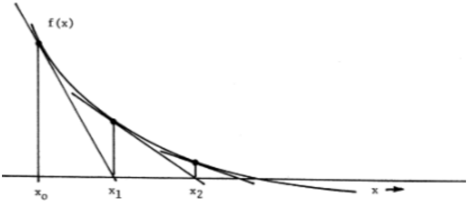
$$\varphi_i(x) = \begin{cases} \frac{x - x_{i-1}}{h}, & \text{if } x \in [x_{i-1}, u_i], \\ \frac{x_{i+1} - x}{h}, & \text{if } x \in [u_i, x_{i+1}], \\ 0, & \text{otherwise.} \end{cases}$$



Finite element method

- Galerkin approximation of the variational formulation: Find $u_h \in V_h$ such that
$$a(u_h, v) = (f, v), \quad v \in V_h.$$
- We express u_h with the nodal basis:
$$u_h(x) = \sum_{i=1}^{N} u_i \varphi_i(x) \quad \text{with} \quad u_i = u_h(u_i).$$
- We obtain an equivalent linear algebraic system:
$$\sum_{i=1}^{N} u_i a(\varphi_i, \varphi_j) = (f, \varphi_j), \quad j = 1, 2, \ldots, N.$$
Equivalently,
$$A_h u_h = f_h, \quad \text{where} \quad A_h = [a(\varphi_j, \varphi_i)]_{i,j=1}^{N},\ f_h = [(f, \varphi_j)]_{j=1}^{N}.$$

An iterative method is a procedure that starts from an initial guess and generates a sequence of increasingly accurate approximations to the solution of a problem.



Given x_0,

$x_{i+1} = x_i - f(x_i)/f'(x_i);\ i = 0, 1, 2, \ldots$

When solving the discrete system $A_h u_h = f_h$:

- **Naive Gaussian elimination (direct method):**

  $\mathcal{O}(N^3)$ computational cost

- **Multigrid method (iterative method):**

  $\mathcal{O}(N)$ computational cost (optimal)

Iterative methods are at the heart of scientific computing.

Jacobi method

- For a 3×3 linear system, the Jacobi iteration updates as:

$$\begin{aligned} a_{11} u_1^{m+1} + a_{12} u_2^{m} + a_{13} u_3^{m} &= f_1, \\ a_{21} u_1^{m} + a_{22} u_2^{m+1} + a_{23} u_3^{m} &= f_2, \\ a_{31} u_1^{m} + a_{32} u_2^{m} + a_{33} u_3^{m+1} &= f_3. \end{aligned}$$

- Each component is updated using only values from the previous iteration.

Gauss–Seidel method

- Improves Jacobi by using the <u>most recent</u> values:

$$\begin{aligned} a_{11} u_1^{m+1} + a_{12} u_2^{m} + a_{13} u_3^{m} &= f_1, \\ a_{21} u_1^{m+1} + a_{22} u_2^{m+1} + a_{23} u_3^{m} &= f_2, \\ a_{31} u_1^{m+1} + a_{32} u_2^{m+1} + a_{33} u_3^{m+1} &= f_3. \end{aligned}$$

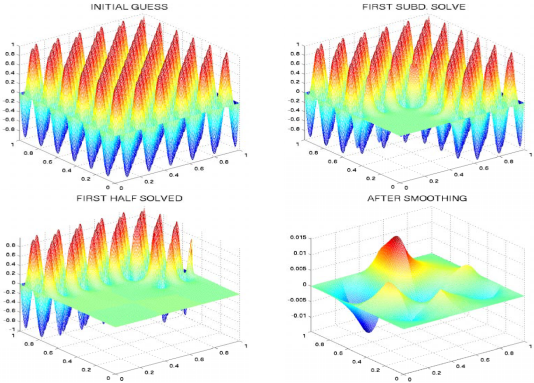
Domain decomposition method

- Domain decomposition $\Rightarrow$ space decomposition:

$$\Omega = \bigcup_{i=1}^{J} \Omega_i, \qquad V_h = \sum_{i=1}^{J} V_i,$$

  where each V_i is a local subspace.

- Illustration of effects of local corrections

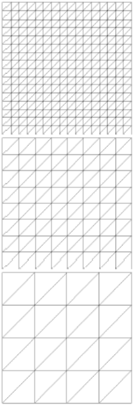


Multigrid method

- Builds a hierarchy of nested spaces:

$$V_h \supset V_{2h} \supset V_{4h} \supset \cdots \supset V_H.$$

- At each level, apply smoothing (e.g., Gauss–Seidel), then transfer residuals to coarser grids for correction.



$$\mathcal{O}(N_h) + \mathcal{O}(N_{2h}) + \mathcal{O}(N_{4h}) + \cdots = \mathcal{O}(N_h)$$

$V_h \Rightarrow (GS)_h \quad \mathcal{O}(N_h)$

$+ \quad \searrow$

$V_{2h} \Rightarrow (GS)_{2h} \quad \mathcal{O}(N_{2h})$

$+ \quad \searrow$

$V_{4h} \Rightarrow (GS)_{4h} \quad \mathcal{O}(N_{4h})$

$+ \quad \searrow$

$V_{8h} \quad \ldots$

Table of Contents

An abstract iterative method for solving $Au = f$ consists of three steps:

1. **Compute the residual**: $r^{m-1} = f - Au^{m-1}$.
2. **Approximate the error**: solve $Ae = r^{m-1}$ approximately, i.e.,

$$\hat{e}^m = Br^{m-1}.$$

3. **Update the solution**: $u^m = u^{m-1} + \hat{e}^m$.

The iterative method is expressed as:

$$u^m = u^{m-1} + B(f - Au^{m-1}), \quad m \geq 1, \tag{Iter}$$

where $B \colon V \to V$ is a linear operator that serves as an approximate inverse of A.

In the form

$$u^m = u^{m-1} + B(b - Au^{m-1}), \quad m \geq 1, \qquad \text{(Iter)}$$

we have

$$B = \begin{cases} \omega I & \text{Richardson,} \\ D^{-1} & \text{Jacobi,} \\ (D+L)^{-1} & \text{Gauss–Seidel,} \end{cases}$$

where

- D is the diagonal part of A,
- L is the strictly lower triangular part,
- U is the strictly upper triangular part.

Two key factors when selecting B:

1. **Accuracy**: B should provide an accurate approximation of A^{-1}.
   - Most accurate choice: $B = A^{-1}$ (fastest one-step convergence but computationally expensive).
2. **Computational cost**: B must be computationally efficient.
   - Least expensive choice: $B = I$ (minimal computational cost but slow convergence).
3. An effective choice of B balances accuracy and computational cost.

- An iterative method $\{u^m\}$ is said to be convergent if

$$\lim_{m\to\infty} u^m = u \quad \text{for any} \quad u^0 \in V.$$

- Error propagation of (Iter):

$$u - u^m = (I - BA)(u - u^{m-1}), \quad m \geq 1.$$

Lemma 2

The iterative method (Iter) *is convergent if and only if*

$$\rho(I - BA) := \max_{\lambda\in\sigma(I-BA)} |\lambda| < 1.$$

Corollary 3

If the iterative method (Iter) *is convergent, then the operator B is nonsingular.*

- We consider the symmetrized iterative scheme:

$$\left.\begin{aligned} u^{m-\frac{1}{2}} &= u^{m-1} + B(f - Au^{m-1}), \\ u^{m} &= u^{m-\frac{1}{2}} + B^t(f - Au^{m-\frac{1}{2}}), \end{aligned}\right\} \quad m \geq 1. \qquad \text{(SymIter)}$$

- This is equivalent to:

$$u^m = u^{m-1} + \bar{B}(f - Au^{m-1}), \quad m \geq 1,$$

where

$$\bar{B} = B^t + B - B^t AB,$$

which is called the symmetrized operator.

- Example: Symmetrized Gauss–Seidel method

$$\begin{aligned} B &= (D+L)^{-1}, \\ \bar{B} &= (D+L)^{-t} D (D+L)^{-1}. \end{aligned}$$

Theorem 4 (Convergence of the abstract iterative method)

For an SPD linear system, the following are equivalent:

(i) *The symmetrized iterative method* (SymIter) *converges.*

(ii) *The operator* $\bar{B} = B^t + B - B^t AB$ *is SPD.*

(iii) *The operator* B *is nonsingular and* $\hat{D} = B^{-1} + B^{-t} - A$ *is SPD.*

Furthermore, if any of these conditions hold, we have

$$\|I - BA\|_A^2 = \lambda_{\max}(I - \bar{B}A) = 1 - \left(\sup_{\|v\|_A = 1} (\bar{B}^{-1} v, v) \right)^{-1}.$$

Corollary 5

The iterative method (Iter) *converges if its symmetrized version* (SymIter) *converges.*

[1] J. Xu and L. Zikatanov. Algebraic multigrid method (Acta Numer. 2017).

Proof of Theorem 4.

- By direct calculation, we have

$$\|(I - BA)v\|_A^2 = ((I - \bar{B}A)v, v)_A = \|v\|_A^2 - (\bar{B}Av, v)_A.$$

- It follows that

$$\|I - BA\|_A^2 = \sup_{\|v\|_A = 1} \|(I - BA)v\|_A^2 = \sup_{\|v\|_A = 1} ((I - \bar{B}A)v, v)_A$$
$$= \lambda_{\max}(I - \bar{B}A) = 1 - \lambda_{\min}(\bar{B}A).$$

- Therefore,

$$\rho(I - BA)^2 \leq \|I - BA\|_A^2 = \rho(I - \bar{B}A) = 1 - \lambda_{\min}(\bar{B}A),$$

which implies (i) $\Leftrightarrow$ (ii).

- Since (ii) implies B is nonsingular, the identity

$$\bar{B} = B^t(B^{-t} + B^{-1} - A)B = B^t \hat{D} B$$

gives the equivalence (ii) $\Leftrightarrow$ (iii).

Proof of Theorem Iter (continued).

- Assuming any of (i)–(iii) holds:

$$\lambda_{\min}(\bar{B}A) > 0 \Rightarrow \lambda_{\min}(\bar{B}A) = [\lambda_{\max}((\bar{B}A)^{-1})]^{-1}.$$

- We obtain

$$\lambda_{\max}((\bar{B}A)^{-1}) = \sup_{\|v\|_A=1} ((\bar{B}A)^{-1}v, v)_A = \sup_{\|v\|_A=1} (\bar{B}^{-1}v, v).$$

- Therefore, we conclude

$$\|I - BA\|_A^2 = 1 - \lambda_{\min}(\bar{B}A) = 1 - \left(\sup_{\|v\|_A=1} (\bar{B}^{-1}v, v) \right)^{-1},$$

which completes the proof.

□

Equivalent characterizations

Theorem 6

The following are equivalent:

(i) *The symmetrized iterative method converges.*

(ii) *The operator* $\bar{B} = B + B^t - B^t AB$ *is SPD.*

(iii) *The operator* B *is nonsingular and the operator* $\hat{D} = B^{-1} + B^{-t} - A$ *is SPD.*

(iv) *The operator* B *is nonsingular and there exists a constant* $\omega \in (0, 2)$ *such that*

$$\left(\frac{2}{\omega} - 1\right)(Av, v) \leq (\hat{D}v, v) \quad \forall v \in V.$$

(v) *The operator* B *is nonsingular and there exists a constant* $\omega \in (0, 2)$ *such that*

$$(2 - \omega)(Bv, v) \leq (\bar{B}v, v) \quad \forall v \in V.$$

(vi) *The operator* B *is nonsingular and there exists a constant* $\omega \in (0, 2)$ *such that*

$$(Av, v) \leq \omega(B^{-1}v, v) \quad \forall v \in V.$$

(vii) *The operator* B *is nonsingular and there exists a constant* $\omega \in (0, 2)$ *such that*

$$(BAv, BAv)_A \leq \omega(BAv, v)_A \quad \forall v \in V.$$

Table of Contents

Richardson iteration

- Simplest iterative method with

$$B = \omega I, \quad \text{where} \quad \omega > 0.$$

- The Richardson iteration is given by

$$u^m = u^{m-1} + \omega(f - Au^{m-1}), \quad m = 1, 2, \ldots .$$

- The method converges if and only if

$$0 < \omega < \frac{2}{\rho(A)}.$$

Gradient descent method

- Consider the SPD linear system $Au = f$ with an equivalent optimization formulation:

$$\min_{v \in V} \left\{ J(v) := \frac{1}{2}(Av, v) - (f, v) \right\}.$$

- Gradient descent updates in the direction of negative gradient:

$$u^m = u^{m-1} - \omega \nabla J(u^{m-1}), \quad m \geq 1.$$

- Since $\nabla J(v) = Av - f$, the method becomes:

$$u^m = u^{m-1} - \omega(Au^{m-1} - f), \quad m \geq 1.$$

- This is identical to Richardson iteration.

Jacobi method

- For a 3×3 system, the Jacobi method is given by

$$\begin{aligned} a_{11} u_1^m + a_{12} u_2^{m-1} + a_{13} u_3^{m-1} &= f_1 \\ a_{21} u_1^{m-1} + a_{22} u_2^m + a_{23} u_3^{m-1} &= f_2 \\ a_{31} u_1^{m-1} + a_{32} u_2^{m-1} + a_{33} u_3^m &= f_3 \end{aligned}$$

- Updates each component by solving the corresponding equation with old data.

- General form: For $i = 1, \ldots, n$

$$u_i^m = u_i^{m-1} + a_{ii}^{-1} \left(f_i - \sum_{j=1}^{n} a_{ij} u_j^{m-1} \right).$$

Gauss–Seidel method

- Improves Jacobi by using the most current estimates:

$$
\begin{aligned}
a_{11} u_1^m + a_{12} u_2^{m-1} + a_{13} u_3^{m-1} &= f_1 \\
a_{21} u_1^m + a_{22} u_2^m + a_{23} u_3^{m-1} &= f_2 \\
a_{31} u_1^m + a_{32} u_2^m + a_{33} u_3^m &= f_3
\end{aligned}
$$

- General form: For $i = 1, \ldots, n$

$$
u_i^m = u_i^{m-1} + a_{ii}^{-1} \left(f_i - \sum_{j=1}^{i-1} a_{ij} u_j^m - \sum_{j=i}^{n} a_{ij} u_j^{m-1} \right).
$$

Matrix splitting formulation

- We split the matrix $A = D + L + U$, where
  - D is the diagonal part of A,
  - L is the strictly lower triangular part,
  - U is the strictly upper triangular part.
- The Jacobi and Gauss–Seidel methods are written as:

$$
\begin{aligned}
Du^m + (L+U)u^{m-1} &= f \quad \text{(Jacobi)},\\
(D+L)u^m + Uu^{m-1} &= f \quad \text{(Gauss–Seidel)}.
\end{aligned}
$$

- In the form $u^m = u^{m-1} + B(f - Au^{m-1})$, we have

$$
B = \begin{cases} D^{-1} & \text{Jacobi},\\ (D+L)^{-1} & \text{Gauss–Seidel}. \end{cases}
$$

Theorem 7 (Convergence of the Jacobi and Gauss–Seidel methods)

Assume that A is SPD. Then we have the following:

- *The Jacobi method converges if and only if $2D - A$ is SPD.*
- *The Gauss–Seidel method always converges. Moreover, we have*

$$\|I - BA\|_A^2 = 1 - \frac{1}{c_1} = 1 - \frac{1}{1 + c_0},$$

  where

$$c_1 = \sup_{v \neq 0} \frac{((D+L)D^{-1}(D+L^t)v, v)}{(v, v)_A}, \quad c_0 = \sup_{v \neq 0} \frac{((LD^{-1}L^t)v, v)}{(v, v)_A}.$$

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[1] J. Xu and L. Zikatanov. The method of alternating projections and the method of subspace corrections in Hilbert space (J. Amer. Math. Soc. 2002).

[2] J. Xu and L. Zikatanov. Algebraic multigrid method (Acta Numer. 2017).

Table of Contents

Optimization formulations of linear systems

Proposition 8

Let $A\colon V \to V$ be a SPD linear operator, and $f \in V$. Given $u \in V$, it is a solution to the linear system

$$Au = f$$

if and only if it solves the quadratic optimization problem

$$\min_{v \in V} \left\{ J(v) := \frac{1}{2}(Av, v) - (f, v) \right\}.$$

Line search and steepest descent

- Line search determines an optimal step size along the search direction p^{m-1} by minimizing the energy functional:

$$\omega_{m-1} = \arg\min_{\omega \in \mathbb{R}} J(u^{m-1} + \omega p^{m-1}).$$

- For quadratic J, the minimizer is:

$$\omega_m = \frac{(r^m, p^m)}{(Ap^m, p^m)}, \quad m \geq 0,$$

where $r^m = f - Au^m$.

- The steepest descent method combines gradient descent with line search:

$$p^{m-1} = -\nabla J(u^{m-1}) = r^{m-1},$$
$$u^m = u^{m-1} + \frac{(r^{m-1}, p^{m-1})}{(Ap^{m-1}, p^{m-1})} p^{m-1}.$$

Steepest descent method

Steepest descent method: locally optimized gradient descent method

Algorithm 1 Steepest descent method

Given $u^0 \in V$
$r^0 = f - Au^0$
for $m = 1, 2, \dots$ **do**
  $\omega_{m-1} = \frac{(r^{m-1}, r^{m-1})}{(Ar^{m-1}, r^{m-1})}$
  $u^m = u^{m-1} + \omega_{m-1} r^{m-1}$
  $r^m = r^{m-1} - \omega_{m-1} A r^{m-1}$
end for

Theorem 9

The steepest descent method satisfies:

$$\|u - u^m\|_A \leq \left(\frac{\kappa(A) - 1}{\kappa(A) + 1} \right) \|u - u^{m-1}\|_A, \quad m \geq 1.$$

Proof.

- It follows that

$$\|u-u^m\|_A = \min_{\alpha\in\mathbb{R}} \|(I-\alpha A)(u-u^{m-1})\|_A \leq \left(\min_{\alpha\in\mathbb{R}} \rho(I-\alpha A)\right) \|u-u^{m-1}\|_A.$$

- Note that

$$\rho(I-\alpha A) = \max_{\lambda\in\sigma(A)} |1-\alpha\lambda| = \max\{1-\alpha\lambda_{\min}(A), -1+\alpha\lambda_{\max}(A)\}.$$

- Hence, $\rho(I-\alpha A)$ is minimized when

$$\alpha = \frac{2}{\lambda_{\min}(A)+\lambda_{\max}(A)},$$

attaining the minimum value

$$\rho(I-\alpha A) = \frac{\lambda_{\max}(A)-\lambda_{\min}(A)}{\lambda_{\max}(A)+\lambda_{\min}(A)} = \frac{\kappa(A)-1}{\kappa(A)+1}.$$

□

Krylov spaces: A systematic way of achieving mutually A-orthogonal search directions

$$\mathcal{K}_0 = \{0\}, \quad \mathcal{K}_m = \mathrm{span}\{p_0, Ap_0, \ldots, A^{m-1}p_0\}, \quad m = 1, 2, \ldots$$

- $\mathcal{K}_0 \subseteq \mathcal{K}_1 \subseteq \ldots$

Conjugate gradient method Find $u_m \in \mathcal{K}_m$ such that

$$(Au_m, v) = (f, v), \quad v \in \mathcal{K}_m.$$

Algorithm 2 Conjugate gradient method

Given $u_0 \in V$; $r_0 = f - Au_0$; $p_0 = r_0$
for $m = 1, 2, \dots$ **do**
  $\alpha_m = \frac{(r_{m-1}, r_{m-1})}{(Ap_{m-1}, p_{m-1})}$
  $u_m = u_{m-1} + \alpha_m p_{m-1}$
  $r_m = r_{m-1} - \alpha_m A p_{m-1}$
  $\beta_m = \frac{(r_m, r_m)}{(r_{m-1}, r_{m-1})}$
  $p_m = r_m + \beta_m p_{m-1}$
end for

Computational cost per iteration: Only one operator-vector multiplication and two inner products

Lemma 10

If u_m is the mth iterate of the conjugate gradient method, then we have the following:

- $u_m - u_0 \in \mathcal{K}_m$ *and* $(Au_m, v) = (f, v)$ *for all* $v \in \mathcal{K}_m$.
- $\|u - u_m\|_A = \inf_{v_m \in u_0 + \mathcal{K}_m} \|u - v_m\|_A$.

Theorem 11

The conjugate gradient method satisfies:

$$\|u - u_m\|_A \leq 2 \left(\frac{\sqrt{\kappa(A)} - 1}{\sqrt{\kappa(A)} + 1} \right)^k m \|u - u_0\|_A.$$

Proof of Theorem 11 can be done by using the Chebyshev polynomials.

The conjugate gradient method still convergences even when we do not have a positive lower bound on the eigenvalues of A, i.e., A is ill-conditioned.

Theorem 12

The conjugate gradient method satisfies:

$$\|u - u_m\|_A^2 \leq \frac{\lambda_{\max}(A)\|u_0 - u\|^2}{(m+1)^2}.$$

Proof of Theorem 12 can be done by using the Jacobi polynomials.

- A preconditioner $B\colon V \to V$ is a SPD linear operator.
- We transform the system $Au = f$ into the preconditioned system:

$$BAu = Bf.$$

  - This system is SPD with respect to the $(\cdot,\cdot)_{B^{-1}}$-inner product.
- We construct B that approximates A^{-1} so that

$$\kappa(BA) \ll \kappa(A).$$

- Extreme choices:
  - $B = A^{-1}$: Smallest possible condition number (1), but difficult to compute.
  - $B = I$: Easy to compute, but no improvement in conditioning.
- A good preconditioner balances approximation quality and computational efficiency.

Preconditioned conjugate gradient method

Preconditioned conjugate gradient method: Conjugate gradient method for solving the preconditioned system $BAu = Bf$ using the $(\cdot,\cdot)_{B^{-1}}$-inner product

Algorithm 3 Preconditioned conjugate gradient method

Given $u_0 \in V$; $r_0 = f - Au_0$; $p_0 = Br_0$
for $m = 1, 2, \dots$ **do**
$\alpha_m = \frac{(r_{m-1}, Br_{m-1})}{(Ap_{m-1}, p_{m-1})}$
$u_m = u_{m-1} + \alpha_m p_{m-1}$
$r_m = r_{m-1} - \alpha_m A p_{m-1}$
$\beta_m = \frac{(r_m, Br_m)}{(r_{m-1}, Br_{m-1})}$
$p_m = Br_m + \beta_m p_{m-1}$
end for

Theorem 13

The preconditioned conjugate gradient method satisfies:

$$\|u - u_m\|_A \le 2\left(\frac{\sqrt{\kappa(BA)} - 1}{\sqrt{\kappa(BA)} + 1}\right)^m \|u - u_0\|_A.$$

- **Abstract Theory of Iterative Methods**
  - Iterative method for solving the SPD linear system $Au = f$:
  $$u^m = u^{m-1} + B(f - Au^{m-1}), \quad m \geq 1.$$
  - The method converges if the symmetrized method converges, and we have
  $$\|I - BA\|_A^2 = 1 - \left(\sup_{\|v\|_A = 1} (\bar{B}^{-1} v, v) \right)^{-1}.$$
  - The convergence theorem can be applied to analyze the Richardson, Jacobi, and Gauss–Seidel methods.
- **Steepest descent and conjugate gradient methods**
  - The steepest descent method satisfies
  $$\|u - u^m\|_A \leq \left(\frac{\kappa(A) - 1}{\kappa(A) + 1} \right) \|u - u^{m-1}\|_A, \quad m \geq 1.$$
  - The conjugate gradient method satisfies
  $$\|u - u^m\|_A \leq 2 \left(\frac{\sqrt{\kappa(A)} - 1}{\sqrt{\kappa(A)} + 1} \right)^m \|u - u^0\|_A, \quad m \geq 1.$$
  - B-preconditioned conjugate gradient method = Conjugate gradient method for solving $BAu = Bf$ using the $(\cdot, \cdot)_{B^{-1}}$-inner product

Chapter 2. Auxiliary Space Theory

Table of Contents

- **Key idea**: A sophisticated iterative method for solving a linear system is interpreted as an elementary iterative method for a larger system, called the auxiliary system.

- **Example**: Domain decomposition and multigrid methods
  $\equiv$ Block Jacobi and Gauss–Seidel methods for the auxiliary system

- The idea of auxiliary space theory can be traced back to Xu (1996)[1].
  - Algorithm design perspective: Hiptmair and Xu (2007)[2]
  - Theoretical analysis perspective: Xu and Zikatanov (2017)[3]

[1] J. Xu. The auxiliary space method and optimal multigrid preconditioning techniques for unstructured grids (Computing 1996).

[2] R. Hiptmair and J. Xu. Nodal auxiliary space preconditioning in $H(\text{curl})$ and $H(\text{div})$ spaces (SIAM J. Numer. Anal. 2007).

[3] J. Xu and L. Zikatanov. Algebraic multigrid methods (Acta Numer. 2017).

Auxiliary system

- We start with the linear system on a Euclidean space V:

$$Au = f, \qquad A \colon V \to V \text{ SPD}, \ f \in V. \tag{Ori}$$

- Introduce another Euclidean space $\utilde{V}$, called the **auxiliary space**, with $\dim \utilde{V} \geq \dim V$.

- Let $\Pi \colon \utilde{V} \to V$ be a surjective linear operator.

- Define the **auxiliary system**:

$$\utilde{A}\,\utilde{u} = \utilde{f}, \tag{Aux}$$

with

$$\utilde{A} = \Pi^t A \Pi \colon \utilde{V} \to \utilde{V}, \qquad \utilde{f} = \Pi^t f \in \utilde{V}.$$

Proposition 14 (Equivalence of systems)

The two linear systems (Ori) *and* (Aux) *are equivalent in the following sense:*

- *If u solves* (Ori) *and $\Pi \utilde{u} = u$, then $\utilde{u}$ solves* (Aux).
- *Conversely, if $\utilde{u}$ solves* (Aux) *and $\Pi \utilde{u} = u$, then u solves* (Ori).

Lemma 15 (Auxiliary space lemma)

Let $\underset{\sim}{B}\colon \underset{\sim}{V} \to \underset{\sim}{V}$ *be a SPD linear operator, and let*

$$B := \Pi \underset{\sim}{B} \Pi^t \colon V \to V.$$

Then, B is SPD, and it satisfies

$$(B^{-1}v, v) = \inf_{\underset{\sim}{v} \in \underset{\sim}{V},\ \Pi \underset{\sim}{v} = v} (\underset{\sim}{B}^{-1} \underset{\sim}{v}, \underset{\sim}{v}), \quad v \in V.$$

[1] J. Xu. Iterative methods by space decomposition and subspace correction (SIAM Rev., 1992).
[2] L. Chen. Deriving the X–Z identity from auxiliary space method (DD21, 2011).
[3] J. Xu and L. Zikatanov. Algebraic multigrid methods (Acta Numer. 2017).

Proof.

- For any $w \in V$, $Bw = 0$ implies

$$0 = (Bw, w) = (\Pi \underset{\sim}{B} \Pi^t w, w) = (\underset{\sim}{B} \Pi^t w, \Pi^t w).$$

- Since $\underset{\sim}{B}$ is SPD, $\Pi^t w = 0$, which implies $w = 0$ by the injectivity of Π^t.
- Hence B is SPD.
- Take any $v \in V$, and define

$$\bar{\underset{\sim}{v}} := \underset{\sim}{B} \Pi^t B^{-1} v \in \underset{\sim}{V}.$$

- Then $\Pi \bar{\underset{\sim}{v}} = v$, and for any $\underset{\sim}{w} \in \underset{\sim}{V}$,

$$(\underset{\sim}{B}^{-1} \bar{\underset{\sim}{v}}, \underset{\sim}{w}) = (\Pi^t B^{-1} v, \underset{\sim}{w}) = (B^{-1} v, \Pi \underset{\sim}{w}).$$

Proof (continued).

- For any $\underset{\sim}{v} \in \underset{\sim}{V}$, $\Pi \underset{\sim}{v} = v$ if and only if

$$\underset{\sim}{v} = \underset{\sim}{\bar{v}} + \underset{\sim}{w}, \quad \text{with } \underset{\sim}{w} \in \underset{\sim}{V},\ \Pi \underset{\sim}{w} = 0.$$

- Therefore,

$$\begin{aligned}(\underset{\sim}{B}^{-1}\underset{\sim}{v}, \underset{\sim}{v}) &= (\underset{\sim}{B}^{-1}(\underset{\sim}{\bar{v}} + \underset{\sim}{w}), \underset{\sim}{\bar{v}} + \underset{\sim}{w}) \\ &= (\underset{\sim}{B}^{-1}\underset{\sim}{\bar{v}}, \underset{\sim}{\bar{v}}) + (\underset{\sim}{B}^{-1}\underset{\sim}{w}, \underset{\sim}{w}) \\ &= (B^{-1}v, v) + (\underset{\sim}{B}^{-1}\underset{\sim}{w}, \underset{\sim}{w}),\end{aligned}$$

where the last two equalities follow from the previous slide.

- Taking the infimum over $\underset{\sim}{v}$ completes the proof.

□

- **Original iterative method** for solving $Au = f$:

$$u^{m+1} = u^m + B(f - Au^m), \qquad m \geq 1. \tag{OriIter}$$

- **Auxiliary iterative method** for solving $\underset{\sim}{A}\underset{\sim}{u} = \underset{\sim}{f}$:

$$\underset{\sim}{u}^{m+1} = \underset{\sim}{u}^m + \underset{\sim}{B}(\underset{\sim}{f} - \underset{\sim}{A}\underset{\sim}{u}^m), \qquad m \geq 1, \tag{AuxIter}$$

with the relation

$$B = \Pi \underset{\sim}{B} \Pi^t.$$

Proposition 16 (Equivalence of iterative methods)

The two iterations (OriIter) *and* (AuxIter) *are equivalent in the following sense:*

- *If* $\{\underset{\sim}{u}^m\}$ *is generated by* (AuxIter)*, then* $\{u^m = \Pi\underset{\sim}{u}^m\}$ *satisfies* (OriIter)*.*
- *Conversely, if* $\{u^m\}$ *is generated by* (OriIter)*, then there exists* $\{\underset{\sim}{u}^m\}$ *satisfying* (AuxIter) *with* $u^m = \Pi\underset{\sim}{u}^m$*.*

Recall that the error propagation operator of the iterative method

$$u^m = u^{m-1} + B(f - Au^{m-1}), \quad m \geq 1,$$

is $I - BA$, i.e.,

$$u - u^m = (I - BA)(u - u^{m-1}), \quad m \geq 1.$$

Proposition 17

In the iterative methods, we have

$$\|I - BA\|_A = |\underset{\sim}{I} - \underset{\sim}{B}\underset{\sim}{A}|_{\underset{\sim}{A}}.$$

Proof.

Since Π is surjective, we get

$$\begin{aligned}
\|I - BA\|_A^2 &= \sup_{v \in V,\ \|v\|_A \neq 0} \frac{(v,v)_A - ((B + B^t - B^t AB)Av, v)_A}{(v,v)_A} \\
&= \sup_{\underset{\sim}{v} \in \underset{\sim}{V},\ |\Pi \underset{\sim}{v}|_A \neq 0} \frac{(\Pi\underset{\sim}{v}, \Pi\underset{\sim}{v})_A - ((B + B^t - B^t AB)A\Pi\underset{\sim}{v}, \Pi\underset{\sim}{v})_A}{(\Pi\underset{\sim}{v}, \Pi\underset{\sim}{v})_A} \\
&= \sup_{\underset{\sim}{v} \in \underset{\sim}{V},\ |\underset{\sim}{v}|_{\underset{\sim}{A}} \neq 0} \frac{(\underset{\sim}{v}, \underset{\sim}{v})_{\underset{\sim}{A}} - ((\underset{\sim}{B} + \underset{\sim}{B}^t - \underset{\sim}{B}^t \underset{\sim}{A}\underset{\sim}{B})\underset{\sim}{A}\underset{\sim}{v}, \underset{\sim}{v})_{\underset{\sim}{A}}}{(\underset{\sim}{v}, \underset{\sim}{v})_{\underset{\sim}{A}}} = |\underset{\sim}{I} - \underset{\sim}{B}\underset{\sim}{A}|_{\underset{\sim}{A}}^2.
\end{aligned}$$

Theorem 18 (Error propagation in terms of the auxiliary space)

If the symmetrized operator

$$\bar{\underset{\sim}{B}} := \underset{\sim}{B} + \underset{\sim}{B}^t - \underset{\sim}{B}^t \underset{\sim}{A} \underset{\sim}{B} \colon \underset{\sim}{V} \to \underset{\sim}{V}$$

is SPD, then (OriIter) *is convergent. Moreover, the* *error propagation operator* $I - BA$ *satisfies*

$$\|I - BA\|_A^2 = 1 - \left(\sup_{v \in V,\ \|v\|_A = 1} \inf_{\underset{\sim}{v} \in \underset{\sim}{V},\ \Pi \underset{\sim}{v} = v} (\bar{\underset{\sim}{B}}^{-1} \underset{\sim}{v}, \underset{\sim}{v}) \right)^{-1} < 1.$$

- In many cases, the original method is structurally complex, while the auxiliary method is relatively simple.
- Auxiliary space theory simplifies analysis by interpreting the original method as a simple iteration on the auxiliary system.

[1] **JP** and J. Xu. Auxiliary space theory for the analysis of iterative methods for semidefinite linear systems (2025+).

Proof.

- By Lemma 15, B is also SPD.
- Invoking the abstract theory of iterative methods, we have

$$\|I - BA\|_A^2 = 1 - \left(\sup_{v \in V,\ \|v\|_A = 1} (\bar{B}^{-1} v, v) \right)^{-1}.$$

- Since $\bar{B} = \Pi \, \underset{\sim}{\bar{B}} \Pi^t$, applying Lemma 15 gives

$$(\bar{B}^{-1} v, v) = \inf_{\underset{\sim}{v} \in \underset{\sim}{V},\ \Pi \underset{\sim}{v} = v} (\underset{\sim}{\bar{B}}^{-1} \underset{\sim}{v}, \underset{\sim}{v}).$$

- Combining the above two equations completes the proof.

□

Convergence theorem - Singular case

Theorem 19 (Error propagation in terms of the auxiliary space)

Suppose that A is semi-SPD. If the symmetrized operator

$$\bar{\underset{\sim}{B}} := \underset{\sim}{B} + \underset{\sim}{B}^t - \underset{\sim}{B}^t \underset{\sim}{A} \underset{\sim}{B} \colon \underset{\sim}{V} \to \underset{\sim}{V}$$

is SPD, then (OriIter) *is convergent. Moreover, the error propagation operator $I - BA$ satisfies*

$$|I - BA|_A^2 = 1 - \left(\sup_{v \in V,\ \|v\|_A = 1} \inf_{\phi \in N(A)} \inf_{\underset{\sim}{v} \in \underset{\sim}{V},\ \Pi \underset{\sim}{v} = v + \phi} (\bar{\underset{\sim}{B}}^{-1} \underset{\sim}{v}, \underset{\sim}{v}) \right)^{-1} < 1.$$

- The auxiliary space framework extends naturally to the case where A is singular.
- The singularity of A does not deteriorate the convergence rate of the method—it even improves the convergence rate!

[1] **JP** and J. Xu. Auxiliary space theory for the analysis of iterative methods for semidefinite linear systems (2025+).

- When B is SPD, then we can use the B-preconditioned conjugate gradient method to solve $Au = f$.
- Recall that the convergence rate depends on

$$\kappa(BA) = \frac{\lambda_{\max}(BA)}{\lambda_{min}(BA)}.$$

Theorem 20

If $\underset{\sim}{B}$ is SPD, then B is also SPD. Moreover, we have

$$(\lambda_{\min}(BA))^{-1} = \sup_{v \in V, \|v\|_A = 1} \inf_{\underset{\sim}{v} \in \underset{\sim}{V}, \Pi \underset{\sim}{v} = v} (\underset{\sim}{B}^{-1} \underset{\sim}{v}, \underset{\sim}{v}),$$

$$(\lambda_{\max}(BA))^{-1} = \inf_{v \in V, \|v\|_A = 1} \inf_{\underset{\sim}{v} \in \underset{\sim}{V}, \Pi \underset{\sim}{v} = v} (\underset{\sim}{B}^{-1} \underset{\sim}{v}, \underset{\sim}{v}).$$

Consequently:

$$\kappa(BA) = \frac{\sup_{v \in V, \|v\|_A = 1} \inf_{\underset{\sim}{v} \in \underset{\sim}{V}, \Pi \underset{\sim}{v} = v} (\underset{\sim}{B}^{-1} \underset{\sim}{v}, \underset{\sim}{v})}{\inf_{v \in V, \|v\|_A = 1} \inf_{\underset{\sim}{v} \in \underset{\sim}{V}, \Pi \underset{\sim}{v} = v} (\underset{\sim}{B}^{-1} \underset{\sim}{v}, \underset{\sim}{v})}.$$

[1] L. Chen. Deriving the X–Z identity from auxiliary space method (DD21, 2011).

[2] **JP** and J. Xu. Auxiliary space theory for the analysis of iterative methods for semidefinite linear systems (2025+).

Proof.

- Since BA is symmetric with respect to the inner product $(\cdot,\cdot)_A$, we have

$$
\begin{aligned}
(\lambda_{\min}(BA))^{-1} &= \lambda_{\max}((BA)^{-1}) \\
&= \sup_{v\in V,\ \|v\|_A=1} ((BA)^{-1}v, v)_A = \sup_{v\in V,\ \|v\|_A=1} (B^{-1}v, v).
\end{aligned}
$$

- Applying Lemma 15 yields

$$
(B^{-1}v, v) = \inf_{\utilde{v}\in\utilde{V},\ \Pi\utilde{v}=v} (\utilde{B}^{-1}\utilde{v}, \utilde{v}).
$$

- Combining the above two equations completes the proof of the $\lambda_{\min}(BA)$ identity.
- We can prove the $(\lambda_{\max}(BA))$ identity in the same manner.

□

Corollary 21 (Lions lemma)

Suppose that $\underset{\sim}{B}$ is SPD, and that the following hold:

1. *For any $\underset{\sim}{v} \in \underset{\sim}{V}$, we have $\|\Pi \underset{\sim}{v}\|_A \leq \tilde{\mu}_1 \|\underset{\sim}{v}\|_{\underset{\sim}{B}^{-1}}$.*
2. *(Stable decomposition) For any $v \in V$, there exists $\underset{\sim}{v} \in \underset{\sim}{V}$ with $\Pi \underset{\sim}{v} = v$ and $\|\underset{\sim}{v}\|_{\underset{\sim}{B}^{-1}} \leq \tilde{\mu}_0 \|v\|_A$.*

Then we have

$$\kappa(BA) \leq (\tilde{\mu}_0 \tilde{\mu}_1)^2.$$

Corollary 22 (Fictitious space lemma)

If $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are SPD, then we have

$$\kappa(BA) \leq \left(\frac{\sup_{v \in V, \|v\|_A = 1} \inf_{\underset{\sim}{v} \in \underset{\sim}{V}, \Pi \underset{\sim}{v} = v} \|\underset{\sim}{v}\|_{\underset{\sim}{A}}}{\inf_{v \in V, \|v\|_A = 1} \inf_{\underset{\sim}{v} \in \underset{\sim}{V}, \Pi \underset{\sim}{v} = v} \|\underset{\sim}{v}\|_{\underset{\sim}{A}}} \right)^2 \kappa(\underset{\sim}{B}\underset{\sim}{A}).$$

[1] S.V. Nepomnyaschikh. Decomposition and fictitious domains methods for elliptic boundary value problems (DD5, 1992).

Subspace correction methods

- (Xu 1992, Xu and Zikatanov 2002, Lee, Wu, Xu, and Zikatanov 2008)
- Auxiliary space of product type $\underset{\sim}{V} = \prod_{j=1}^{J} V_j$ (Chen 2011)
- Unified analysis for parallel/successive methods for nonsingular/singular problems

Multigrid methods for unstructured grids

- (Xu 1996, Zhang and Xu 2014)
- Equivalent to multigrid methods for auxiliary structure grids

Hiptmair–Xu preconditioners

- Poisson-based optimal preconditioners for $H(\mathrm{curl})$ and $H(\mathrm{div})$ problems (Hiptmair and Xu 2007)
- Equivalent to block Jacobi methods for certain auxiliary systems given in terms of regular decomposition

[1] **JP** and J. Xu. Auxiliary space theory for the analysis of iterative methods for semidefinite linear systems (2025+).

Mixed finite element methods

- Darcy flow, Stokes equations and their generalizations
- Sharp and unified estimates for the Schur complements:

  Auxiliary space theory + inf–sup condition (Xu and Zikatanov 2002)

Nonoverlapping domain decomposition methods

- FETI, FETI-DP, BDD, BDDC (Toselli and Widlund 2005)
- Sharp and unified analysis:

  Auxiliary space theory + Basic lemmas for nonoverlapping domain decomposition (Xu and Zou 1998)

[1] **JP**. Unified analysis of saddle point problems via auxiliary space theory (2025+).

Table of Contents

- **Subspace correction methods** provide a unified framework that encompasses a wide range of algorithms.
  - Classical Jacobi and Gauss–Seidel methods;
  - More advanced multigrid and domain decomposition methods.
- High-level description
  1. The solution space is decomposed into a sum of subspaces.
  2. Local problems are solved independently on each
  3. The local solutions are combined to update the global iterate.
- A sharp convergence theory, known as the Xu–Zikatanov identity, is available.
- Applications to not only SPD linear systems, but also singular, nearly singular, and even nonlinear problems.

[1] J. Xu. Iterative methods by space decomposition and subspace correction (SIAM Rev., 1992).

[2] J. Xu and L. Zikatanov. The method of alternating projections and the method of subspace corrections in Hilbert spaces (J. Amer. Math. Soc. 2002).

[3] B. Jiang, **JP**, and J. Xu. Connections between convex optimization algorithms and subspace correction methods (2025+).

Space decomposition

- Consider subspaces $V_1, V_2, \ldots, V_J \subset V$ with

$$V = \sum_{j=1}^{J} V_j = \sum_{j=1}^{J} I_j V_j, \quad I_j \colon V_j \to V \text{ natural embedding.}$$

- In each V_j, define the local operator

$$A_j = I_j^t A I_j, \quad I_j^t \colon V \to V \text{ orthogonal projection.}$$

Subspace correction[1]

- The optimal global correction e can be obtained by solving the residual equation

$$Ae = r^{\text{old}} (:= f - Au^{\text{old}}).$$

- Instead of the global equation, we consider the <u>local problem</u>:

$$A_j e_j = I_j^t r^{\text{old}}.$$

- We update the global approximation as

$$u^{\text{new}} = u^{\text{old}} + e_j.$$

[1] J. Xu. Iterative methods by space decomposition and subspace correction (SIAM Rev. 1992).

Parallel subspace correction method (PSC)

- Residual is corrected simultaneously across all auxiliary spaces:

$$u^{\text{new}} = u^{\text{old}} + \sum_{j=1}^{J} e_j.$$

- PSC can also be interpreted as an SPD preconditioner.
- Examples: Jacobi, additive Schwarz, BPX preconditioners, ...

Successive subspace correction method (SSC)

- Residual is corrected sequentially, one local space at a time.
- Examples: Gauss–Seidel, multiplicative Schwarz, multigrid cycles, ...

Remark 1 (Extensions)

Subspace correction methods extend to more general settings: when each V_j is not a true subspace of V, one may instead use inexact local problems.[1]

[1] J. Xu and L. Zikatanov. Algebraic multigrid methods (Acta Numer. 2017).

Examples of PSC

Example 23 (Jacobi Method)

- $V = \mathbb{R}^n$ with coordinate-wise decomposition: $V = \sum_{i=1}^n \text{span}\{e_i\}$
- $\Pi_i^t v = (e_i, v)e_i = e_i e_i^t v$
- $A_i = \Pi_i^t A \Pi_i = a_{ii}$
- With $R_i = A_i^{-1} = a_{ii}^{-1}$, PSC reduces to the Jacobi method:

$$u_i^m = u_i^{m-1} + a_{ii}^{-1}\left(f_i - \sum_{j=1}^n a_{ij} u_j^{m-1}\right).$$

Example 24 (Richardson Method)

- Same setting as Jacobi method
- Choose $R_i = \omega$ for some $\omega > 0$.
- PSC reduces to the Richardson iteration

Example 25 (Gauss–Seidel and SOR)

- $V = \mathbb{R}^n$ with coordinate-wise decomposition: $V = \sum_{i=1}^n \operatorname{span}\{e_i\}$
- $\Pi_i^t v = (e_i, v)e_i = e_i e_i^t v$
- $A_i = \Pi_i^t A \Pi_i = a_{ii}$
- With $R_i = A_i^{-1} = a_{ii}^{-1}$, SSC reduces to the Gauss–Seidel method:

$$u_i^m = u_i^{m-1} + a_{ii}^{-1}\left(f_i - \sum_{j<i} a_{ij}u_j^m - \sum_{j>i} a_{ij}u_j^{m-1}\right).$$

- With $R_i = \omega A_i^{-1} = \omega a_{ii}^{-1}$ for some $\omega > 0$, SSC reduces to the SOR method.

Domain decomposition ⇒ Space decomposition

$$\Omega = \bigcup_{i=1}^{J} \Omega_i \Rightarrow V = \sum_{i=1}^{J} V_i,$$

where each V_i is a "local" subspace:

$$V_i = \{v \in V : v(x) = 0, \ \forall x \in \Omega \setminus \Omega_i\} \subset V \equiv H_0^1(\Omega).$$

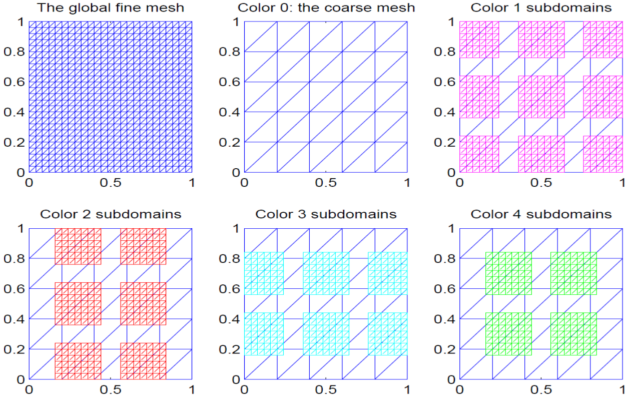
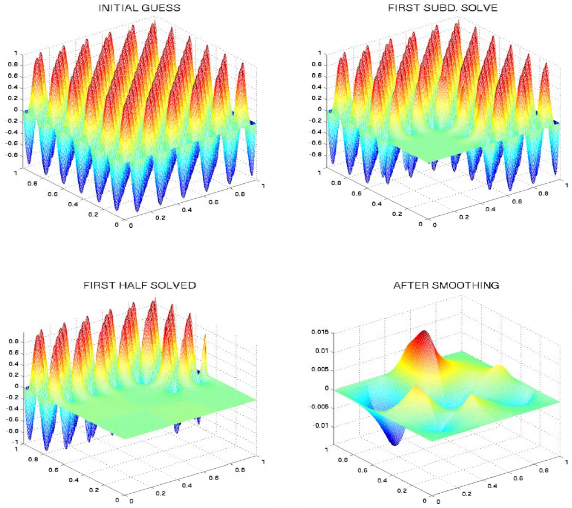


Illustration of Effects of Local Corrections



Product-type auxiliary space

We interpret subspace correction methods in the framework of the auxiliary space theory.

- **Auxiliary space of product type**:

$$\underset{\sim}{V} = \prod_{j=1}^{J} V_j = V_1 \times V_2 \times \cdots \times V_J.$$

- The operator $\Pi \colon \underset{\sim}{V} \to V$ is defined as

$$\Pi \underset{\sim}{u} = \sum_{j=1}^{J} u_j = \sum_{j=1}^{J} I_j u_j.$$

- Auxiliary system:

$$\underset{\sim}{A} \underset{\sim}{u} = \underset{\sim}{f},$$

  where

$$\underset{\sim}{A} = \Pi^t A \Pi = [A_{ij}]_{i,j=1}^{J}, \quad \text{with } A_{ij} = I_i^t A I_j.$$

- Block decomposition:

$$\underset{\sim}{A} = \underset{\sim}{L} + \underset{\sim}{D} + \underset{\sim}{L}^t.$$

Theorem 26 (Subspace correction methods ≡ Block Jacobi/Gauss–Seidel)

- *PSC for $Au = f$ is equivalent to the block Jacobi method for $\underset{\sim}{A}\underset{\sim}{u} = \underset{\sim}{f}$:*

$$B_{PSC} = \Pi \underset{\sim}{B}_{PSC} \Pi^t, \qquad \underset{\sim}{B}_{PSC} = \underset{\sim}{D}^{-1}.$$

- *SSC for $Au = f$ is equivalent to the block Gauss–Seidel method for $\underset{\sim}{A}\underset{\sim}{u} = \underset{\sim}{f}$:*

$$B_{SSC} = \Pi \underset{\sim}{B}_{SSC} \Pi^t, \qquad \underset{\sim}{B}_{SSC} = (\underset{\sim}{L} + \underset{\sim}{D})^{-1}.$$

- Subspace correction methods are equivalent to block methods for solving the auxiliary system.
- Analyzing subspace correction methods is as straightforward as analyzing block Jacobi/Gauss–Seidel methods.

Thanks to the auxiliary space theory, we obtain sharp convergence estimates for subspace correction methods straightforward.

Theorem 27 (Convergence of PSC, additive Schwarz lemma)

In PSC, we have

$$\kappa(B_{PSC}A) = \frac{\displaystyle\sup_{\|v\|_A=1} \inf_{\sum_{j=1}^J v_j = v} \sum_{j=1}^J (A_j v_j, v_j)}{\displaystyle\inf_{\|v\|_A=1} \inf_{\sum_{j=1}^J v_j = v} \sum_{j=1}^J (A_j v_j, v_j)}.$$

[1] J. Xu. Iterative methods by space decomposition and subspace correction (SIAM Rev., 1992).
[2] A. Toselli and O. Widlund. Domain Decomposition Methods—Algorithms and Theory (2005).
[3] S.C. Brenner. An additive analysis of multiplicative Schwarz methods (Numer. Math. 2013).
[4] **JP**. Additive Schwarz methods for convex optimization as gradient methods (SIAM J. Numer. Anal. 2020).

Theorem 28 (Convergence of SSC, Xu–Zikatanov identity)

In SSC, we have

$$\|I - B_{SSC}A\|_A^2 = 1 - \frac{1}{1 + c_0},$$

where

$$c_0 = \sup_{\|v\|_A = 1} \inf_{\sum \Pi_j v_j = v} \sum_{j=1}^{J} \left\| I_j^* \sum_{i>j} v_i \right\|_{A_j}^2 .$$

[1] J. Xu and L. Zikatanov. The method of alternating projections and the method of subspace corrections in Hilbert space (J. Amer. Math. Soc. 2002).

[2] Y.-J. Lee, J. Wu, J. Xu and L. Zikatanov. A sharp convergence estimate for the method of subspace corrections for singular systems of equations (Math. Comp. 2008).

[3] S.C. Brenner. An additive analysis of multiplicative Schwarz methods (Numer. Math. 2013).

Table of Contents

Mixed formulations of second-order elliptic equations

- Poisson equation on a bounded domain $\Omega \subset \mathbb{R}^3$:

$$-\Delta u = f \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

- Introduce $\sigma = -\nabla u$ to rewrite the equation as a first-order system:

$$\sigma + \nabla u = 0, \quad \operatorname{div} \sigma = f.$$

- Mixed variational formulation: Find $(\sigma, u) \in H(\operatorname{div}; \Omega) \times L^2(\Omega)$ such that

$$\begin{aligned} (\sigma, \tau) + (u, \operatorname{div} \tau) &= 0, && \forall \tau \in H(\operatorname{div}; \Omega), \\ (\operatorname{div} \sigma, v) &= (f, v), && \forall v \in L^2(\Omega). \end{aligned}$$

[1] D. Boffi, F. Brezzi, and M. Fortin. Mixed finite element methods and applications (2013).

Maxwell equations

- Coupled system for the electric field E and magnetic field H:

$$\epsilon\frac{\partial E}{\partial t} + \sigma E - \operatorname{curl} H = j \quad \text{in } \Omega \times (0, T),$$
$$\mu\frac{\partial H}{\partial t} + \operatorname{curl} E = 0 \quad \text{in } \Omega \times (0, T).$$

- Eliminating H and discretizing in time: Find $E \in H_0(\operatorname{curl};\Omega)$ such that

$$\left(\frac{1}{4\mu}\Delta t^2 \operatorname{curl} E, \operatorname{curl}\xi\right) + \left(\left(\epsilon + \frac{1}{2}\sigma\Delta\tau\right) E, \xi\right) = f(\xi), \quad \forall\xi \in H_0(\operatorname{curl};\Omega).$$

[1] R. Hiptmair. Multigrid method for Maxwell's equations (SIAM J. Numer. Anal. 1997).

- The operators curl and div have **large, nontrivial kernels**.
  $\Rightarrow$ Efficient and robust solvers are more challenging to design than for scalar elliptic problems.

- The **Hiptmair–Xu preconditioner** provides a general and effective framework for solving problems in $H(\text{curl})$ and $H(\text{div})$.

- The construction is based on the **auxiliary space theory**.

- The method reduces the original system to a sequence of scalar Poisson problems.
  - For example, a 3D $H(\text{curl})$ problem is decomposed into **four scalar Poisson equations**.

[1] R. Hiptmair and J. Xu. Nodal auxiliary space preconditioning in $H(\text{curl})$ and $H(\text{div})$ spaces (SIAM J. Numer. Anal. 2007).

Model problem

- $H(\mathrm{D};\Omega)$ model problem:

$$(\mathrm{D}^*\mathrm{D} + I)\, u = f \quad \text{in } \Omega,$$
$$\operatorname{tr} u = 0 \quad \text{on } \partial\Omega,$$

  where D denotes either the curl or div operator.

- Weak formulation:

$$(\mathrm{D}u, \mathrm{D}v) + (u, v) = (f, v) \quad \forall v \in H_0(\mathrm{D};\Omega).$$

- Finite element discretization using the space $H_h(\mathrm{D})$ yields the linear system:

$$A_{\mathrm{D}}\, u = f.$$

- Functions in $H(\mathrm{curl})$ and $H(\mathrm{div})$ are generally not regular enough to lie in H^1.
- However, they can be decomposed into components that each belong to more regular (e.g., H^1) spaces.
- Such decompositions are known as **regular decompositions**.
- They play a central role in the construction of Hiptmair–Xu preconditioners.
- Regular decompositions exist in both continuous and discrete settings:
  - **Continuous setting:** in terms of Sobolev spaces.
  - **Discrete setting:** in terms of finite element spaces.

[1] R. Hiptmair and J. Xu. Nodal auxiliary space preconditioning in $H(\mathrm{curl})$ and $H(\mathrm{div})$ spaces (SIAM J. Numer. Anal. 2007).

Discrete regular decompositions

Theorem 29 (Discrete regular decomposition for $H_h(\mathrm{curl})$)

For any $v_h \in H_h(\mathrm{curl};\Omega)$, *there exist* $\tilde{v}_h \in H_h(\mathrm{curl};\Omega)$, $\psi_h \in [H_h(\mathrm{grad};\Omega)]^3$, *and* $p_h \in H_h(\mathrm{grad};\Omega)$, *such that*

$$v_h = \tilde{v}_h + \Pi_h^{\mathrm{curl}}\psi_h + \mathrm{grad}\, p_h,$$

$$\|h^{-1}\tilde{v}_h\| + \|\psi_h\|_1 + \|p_h\|_1 \lesssim \|v_h\|_{H(\mathrm{curl})}.$$

Theorem 30 (Discrete regular decomposition for $H_h(\mathrm{div})$)

For any $v_h \in H_h(\mathrm{div};\Omega)$, *there exist* $\tilde{v}_h \in H_h(\mathrm{div};\Omega)$, $\psi_h \in [H_h(\mathrm{grad};\Omega)]^3$, *and* $w_h \in H_h(\mathrm{curl};\Omega)$, *such that*

$$v_h = \tilde{v}_h + \Pi_h^{\mathrm{div}}\psi_h + \mathrm{curl}\, w_h,$$

$$\|h^{-1}\tilde{v}_h\| + \|\psi_h\|_1 + \|w_h\|_1 \lesssim \|v_h\|_{H(\mathrm{div})}.$$

- Motivated by discrete regular decomposition, we consider the space decomposition

$$H_h(\mathrm{curl}) = H_h(\mathrm{curl}) + \Pi_h^{\mathrm{curl}} H_h(\mathrm{grad})^3 + \mathrm{grad}\, H_h(\mathrm{grad}).$$

- Equivalently,

$$V_{\mathrm{curl}} = \Pi_{\mathrm{curl}} \underset{\sim}{V}_{\mathrm{curl}},$$

where

$$\begin{aligned} V_{\mathrm{curl}} &= H_h(\mathrm{curl}), \\ \underset{\sim}{V}_{\mathrm{curl}} &= H_h(\mathrm{curl}) \times H_h(\mathrm{grad})^3 \times H_h(\mathrm{grad}), \\ \Pi_{\mathrm{curl}} &= \left[I, \quad \Pi_h^{\mathrm{curl}}, \quad \mathrm{grad} \right]. \end{aligned}$$

- The Hiptmair–Xu preconditioner B_{curl} is defined by:

$$B_{\mathrm{curl}} = D_{\mathrm{curl}}^{-1} + \Pi_h^{\mathrm{curl}} A_{\mathrm{grad}^3}^{-1} (\Pi_h^{\mathrm{curl}})^t + \mathrm{grad}\, A_{\mathrm{grad}}^{-1}\, \mathrm{grad}^t,$$

where D_{curl}^{-1} is the nodal Jacobi smoother:

$$(v, w)_{D_{\mathrm{curl}}} = \sum_{e \in \mathcal{E}_h} (v_e, w_e)_{H(\mathrm{curl})}, \quad \text{with} \quad v = \sum_{e \in \mathcal{E}_h} v_e, \; w = \sum_{e \in \mathcal{E}_h} w_e.$$

- Equivalently,

$$B_{\mathrm{curl}} = \Pi_{\mathrm{curl}} \underset{\sim}{B}_{\mathrm{curl}} \Pi_{\mathrm{curl}}^t, \quad \underset{\sim}{B}_{\mathrm{curl}} = \begin{bmatrix} D_{\mathrm{curl}}^{-1} & 0 & 0 \\ 0 & A_{\mathrm{grad}^3}^{-1} & 0 \\ 0 & 0 & A_{\mathrm{grad}}^{-1} \end{bmatrix}.$$

[1] R. Hiptmair and J. Xu. Nodal auxiliary space preconditioning in $H(\mathrm{curl})$ and $H(\mathrm{div})$ spaces (SIAM J. Numer. Anal. 2007).

Optimality of Hiptmair–Xu preconditioner for $H(\mathrm{curl})$

Theorem 31

The Hiptmair–Xu preconditioner B_{curl} satisfies

$$\kappa(B_{\mathrm{curl}} A_{\mathrm{curl}}) \lesssim 1.$$

Proof.

- Note that

$$\|\underset{\sim}{v}_h\|^2_{\underset{\sim}{B}_{\mathrm{curl}}^{-1}} = \|\tilde{v}_h\|^2_{D_{\mathrm{curl}}} + \|\psi_h\|^2_{A_{\mathrm{grad}^3}} + \|p_h\|^2_{A_{\mathrm{grad}}}, \quad \underset{\sim}{v}_h = (\tilde{v}_h, \psi_h, p_h) \in \underset{\sim}{V}_{\mathrm{curl}}.$$

- Thanks to the auxiliary space theory, it suffices to verify the following:
  - (a) For any $\tilde{v}_h \in H_h(\mathrm{curl})$, $\psi_h \in H_h(\mathrm{grad})^3$, and $p_h \in H_h(\mathrm{grad})$,

$$\|\tilde{v}_h + \Pi_h^{\mathrm{curl}} \psi_h + \mathrm{grad}\, p_h\|^2_{A_{\mathrm{curl}}} \lesssim \|\tilde{v}_h\|^2_{D_{\mathrm{curl}}} + \|\psi_h\|^2_{A_{\mathrm{grad}^3}} + \|p_h\|^2_{A_{\mathrm{grad}}}.$$

  - (b) For any $v_h \in H_h(\mathrm{curl})$, there exist $\tilde{v}_h \in H_h(\mathrm{curl})$, $\psi_h \in H_h(\mathrm{grad})^3$, and $p_h \in H_h(\mathrm{grad})$ such that

$$v_h = \tilde{v}_h + \Pi_h^{\mathrm{curl}} \psi_h + \mathrm{grad}\, p_h,$$

$$\|\tilde{v}_h\|^2_{D_{\mathrm{curl}}} + \|\psi_h\|^2_{A_{\mathrm{grad}^3}} + \|p_h\|^2_{A_{\mathrm{grad}}} \lesssim \|v_h\|^2_{A_{\mathrm{curl}}}.$$

Proof (continued).

- We first prove part (a). Take any $\tilde{v}_h \in H_h(\mathrm{curl})$, $\psi_h \in H_h(\mathrm{grad})^3$, and $p_h \in H_h(\mathrm{grad})$. By the standard coloring argument, we have the estimate

$$\|\tilde{v}_h\|_{A_{\mathrm{curl}}} \lesssim \|\tilde{v}_h\|_{D_{\mathrm{curl}}}.$$

- From the commutativity and stability of the canonical interpolation operators, we deduce

$$\begin{aligned}\|\Pi_h^{\mathrm{curl}}\psi_h\|_{A_{\mathrm{curl}}}^2 &= \|\Pi_h^{\mathrm{curl}}\psi_h\|^2 + \|\operatorname{curl}\Pi_h^{\mathrm{curl}}\psi_h\|^2 \\ &\lesssim \|\psi_h\|^2 + \|\Pi_h^{\mathrm{div}}\operatorname{curl}\psi_h\|^2 \lesssim \|\psi_h\|^2 + |\psi_h|_{H(\mathrm{grad})^3}^2 = \|\psi_h\|_{A_{\mathrm{grad}^3}}^2.\end{aligned}$$

- Additionally, we directly obtain

$$\|\operatorname{grad} p_h\|_{A_{\mathrm{curl}}}^2 = \|\operatorname{grad} p_h\|^2 \leq \|p_h\|_{A_{\mathrm{grad}}}^2.$$

- Combining the above estimates, we conclude

$$\begin{aligned}\left\|\tilde{v}_h + \Pi_h^{\mathrm{curl}}\psi_h + \operatorname{grad} p_h\right\|_{A_{\mathrm{curl}}}^2 &\lesssim \|\tilde{v}_h\|_{A_{\mathrm{curl}}}^2 + \|\Pi_h^{\mathrm{curl}}\psi_h\|_{A_{\mathrm{curl}}}^2 + \|\operatorname{grad} p_h\|_{A_{\mathrm{curl}}}^2 \\ &\lesssim \|\tilde{v}_h\|_{D_{\mathrm{curl}}}^2 + \|\psi_h\|_{A_{\mathrm{grad}^3}}^2 + \|p_h\|_{A_{\mathrm{grad}}}^2.\end{aligned}$$

Proof (continued).

- Next, we prove part (b). Given $v_h \in H_h(\mathrm{curl})$, let $\tilde{v}_h \in H_h(\mathrm{curl})$, $\psi_h \in H_h(\mathrm{grad})^3$, and $p_h \in H_h(\mathrm{grad})$ be given by the discrete regular decomposition :

$$\|h^{-1}\tilde{v}_h\|^2 + \|\psi_h\|^2_{A^3_{\mathrm{grad}}} + \|p_h\|^2_{A_{\mathrm{grad}}} \lesssim \|v_h\|^2_{A_{\mathrm{curl}}}.$$

- It remains to show

$$\|\tilde{v}_h\|_{D_{\mathrm{curl}}} \lesssim \|h^{-1}\tilde{v}_h\|.$$

- This follows directly from the inverse inequality and the finite overlap property:

$$\begin{aligned}\|\tilde{v}_h\|^2_{D_{\mathrm{curl}}} &= \sum_{e\in\mathcal{E}_h} (\tilde{v}_e, \tilde{v}_e)_{H(\mathrm{curl})} = \sum_{e\in\mathcal{E}_h} \left(\|\,\mathrm{curl}\,\tilde{v}_e\|^2 + \|\tilde{v}_e\|^2\right) \\ &\lesssim \sum_{e\in\mathcal{E}_h} \|h^{-1}\tilde{v}_e\|^2 + \sum_{e\in\mathcal{E}_h} \|\tilde{v}_e\|^2 \lesssim \|h^{-1}\tilde{v}_h\|^2 + \|\tilde{v}_h\|^2 \lesssim \|h^{-1}\tilde{v}_h\|^2.\end{aligned}$$

□

- Motivated by discrete regular decomposition, we consider the space decomposition

$$H_h(\mathrm{div}) = H_h(\mathrm{div}) + \Pi_h^{\mathrm{div}} H_h(\mathrm{grad})^3 + \mathrm{curl}\, H_h(\mathrm{curl}).$$

- Equivalently,

$$V_{\mathrm{div}} = \Pi_{\mathrm{div}} \underset{\sim}{V}_{\mathrm{div}},$$

where

$$\begin{aligned} V_{\mathrm{div}} &= H_h(\mathrm{div}), \\ \underset{\sim}{V}_{\mathrm{div}} &= H_h(\mathrm{div}) \times H_h(\mathrm{grad})^3 \times H_h(\mathrm{curl}), \\ \Pi_{\mathrm{div}} &= \left[I, \quad \Pi_h^{\mathrm{div}}, \quad \mathrm{curl} \right]. \end{aligned}$$

- The Hiptmair–Xu preconditioner B_{div} is defined by:

$$B_{\mathrm{div}} = D_{\mathrm{div}}^{-1} + \Pi_h^{\mathrm{div}} A_{\mathrm{grad}^3}^{-1} (\Pi_h^{\mathrm{div}})^t + \mathrm{curl}\, A_{\mathrm{curl}}^{-1}\, \mathrm{curl}^t,$$

where D_{div}^{-1} is the nodal Jacobi smoother:

$$(v, w)_{D_{\mathrm{div}}} = \sum_{F \in \mathcal{F}_h} (v_F, w_F)_{H(\mathrm{div})}, \quad \text{with} \quad v = \sum_{F \in \mathcal{F}_h} v_F, \; w = \sum_{F \in \mathcal{F}_h} w_F.$$

- Equivalently,

$$B_{\mathrm{div}} = \Pi_{\mathrm{div}} \underset{\sim}{B}_{\mathrm{div}} \Pi_{\mathrm{div}}^t, \quad \underset{\sim}{B}_{\mathrm{div}} = \begin{bmatrix} D_{\mathrm{div}}^{-1} & 0 & 0 \\ 0 & A_{\mathrm{grad}^3}^{-1} & 0 \\ 0 & 0 & A_{\mathrm{curl}}^{-1} \end{bmatrix}.$$

[1] R. Hiptmair and J. Xu. Nodal auxiliary space preconditioning in $H(\mathrm{curl})$ and $H(\mathrm{div})$ spaces (SIAM J. Numer. Anal. 2007).

Theorem 32

The Hiptmair–Xu preconditioner B_{div} satisfies

$$\kappa(B_{\text{div}} A_{\text{div}}) \lesssim 1.$$

- Similar proof technique as for $H(\text{curl})$ case.
- Uses discrete regular decomposition for $H_h(\text{div})$.

- **Auxiliary space theory**
    - A sophisticated (complicated) iterative method for solving a linear system can be equivalent to a simple (elementary) iterative method for solving an auxiliary linear system.
    - The convergence rate can be expressed in terms of the auxiliary space:

$$\|I - BA\|_A^2 = 1 - \left(\sup_{v \in V,\ \|v\|_A = 1} \inf_{\underset{\sim}{v} \in \underset{\sim}{V},\ \Pi \underset{\sim}{v} = v} (\bar{\underset{\sim}{B}}^{-1} \underset{\sim}{v}, \underset{\sim}{v}) \right)^{-1}.$$

- **Subspace correction methods**
    - Subspace correction methods are equivalent to simple block methods for solving the expanded system.
    - The Xu–Zikatanov identity, a sharp convergence rate estimate for SSC, can be proven by using the auxiliary space theory.

$$\|I - BA\|_A^2 = 1 - \frac{1}{1 + c_0} = 1 - \frac{1}{c_1}.$$

- **Hiptmair–Xu preconditioners**
    - Discrete regular decompositions and the auxiliary space theory
    - Optimal preconditioners for $H(\mathrm{curl})$ and $H(\mathrm{div})$ systems:

$$B_{\mathrm{curl}} = D_{\mathrm{curl}}^{-1} + \Pi_h^{\mathrm{curl}} A_{\mathrm{grad}^3}^{-1} (\Pi_h^{\mathrm{curl}})^t + \mathrm{grad}\, A_{\mathrm{grad}}^{-1}\, \mathrm{grad}^t,$$

$$B_{\mathrm{div}} = D_{\mathrm{div}}^{-1} + \Pi_h^{\mathrm{div}} A_{\mathrm{grad}^3}^{-1} (\Pi_h^{\mathrm{div}})^t + \mathrm{curl}\, A_{\mathrm{curl}}^{-1}\, \mathrm{curl}^t\,.$$

Table of Contents

- Let V and W be finite-dimensional vector spaces.
- Consider the saddle point problem

$$\begin{bmatrix} A & B^t \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}, \qquad \text{(Saddle)}$$

where

- $A \colon V \to V$ is SPD, $B \colon V \to W$ is surjective, $f \in V$, $g \in W$
- Babuška–Brezzi (inf–sup) condition

$$\inf_{q \in W,\ \|q\|=1} \sup_{v \in V,\ \|v\|=1} (Bv, q) = \|B^{-1}\|^{-1} > 0$$

- The system is well-posed if $\mathcal{N}(A) \cap \mathcal{N}(B) = \{0\}$.

Proposition 33

The saddle point problem (Saddle) *is equivalent to the constrained minimization problem*

$$\min_{v \in V} \left\{ \frac{1}{2}(Av, v) - (f, v) \right\} \quad \textit{subject to} \quad Bv = g,$$

with p as the Lagrange multiplier.

Schur complement (dual) system

- Eliminating the primal variable u yields

$$Sp = d, \qquad \text{(Schur)}$$

where

$$S = BA^{-1}B^t \colon W \to W, \quad d = BA^{-1}f - g.$$

- S is called **Schur complement**.
- Since B is surjective and A^{-1} is SPD, S is SPD.
- **Crucial observation:** S has the auxiliary space structure

$$S = \underbrace{B}_{\Pi} \underbrace{A^{-1}}_{\underset{\sim}{B}} \underbrace{B^t}_{\Pi^t}.$$

Remark 2 (Semi-SPD case)

A projected Schur complement system can still be constructed even when A is only semi-SPD.[1,2]

[1] C. Farhat and F.X. Roux. A method of finite element tearing and interconnecting and its parallel solution algorithm (Internat. J. Numer. Methods Engrg. 1991).

[2] C. Pechstein. Finite and boundary element tearing and interconnecting solvers for multiscale problems (2012).

Three major approaches[1][2]

- Iterative methods for the Schur complement system
  - Preconditioned conjugate gradient method for solving (Schur).
  - We can utilize vast existing results on iterative methods for SPD linear systems.
- Stationary iterations
  - Uzawa-type and augmented Lagrangian methods.
  - Convergence behavior is determined by properties of the Schur complement.
- Preconditioned Krylov methods
  - MINRES with an optimal block-diagonal preconditioner[3]

$$\begin{bmatrix} A^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix}.$$

The properties of the Schur complement S are central to both design and analysis of algorithms.

[1] M. Benzi, G.H. Golub, and J. Liesen. Numerical solution of saddle point problems (Acta Numer. 2005).

[2] J. Xu. Fast Poisson-based solvers for linear and nonlinear PDEs (Proc. ICM 2010).

[3] M.F. Murphy, G.H. Golub, and A.J. Wathen, A note on preconditioning for indefinite linear systems (SIAM J. Sci. Comput. 2000).

Theorem 34 (Spectrum of the Schur complement)

The Schur complement $S = BA^{-1}B^t$ *satisfies*

$$\lambda_{\min}(S) = \inf_{0 \neq q \in W} \sup_{v \in V,\ Bv = q} \frac{\|q\|^2}{(Av, v)}, \qquad \lambda_{\max}(S) = \sup_{0 \neq v \in V} \frac{\|Bv\|^2}{(Av, v)}.$$

Corollary 35

The Schur complement S *given in* (Schur) *satisfies*

$$\lambda_{\min}(S) \geq \lambda_{\max}(A)^{-1} \|B^{-1}\|^{-2}, \quad \lambda_{\max}(S) \leq \lambda_{\min}(A)^{-1} \|B\|^2.$$

[1] **JP**. Unified analysis of saddle point problems via auxiliary space theory (2025+).

[2] D. Boffi, F. Brezzi, and M. Fortin. Mixed finite element methods and applications (2013).

[3] A. Toselli and O. Widlund. Domain decomposition methods—Algorithms and theory (2005).

Proof of Theorem 34.

- In the auxiliary space theory, we set

$$V \leftarrow W, \quad \underaccent{\tilde}{V} \leftarrow V, \quad A \leftarrow I, \quad B \leftarrow S, \quad \Pi \leftarrow B, \quad \underaccent{\tilde}{B} \leftarrow A^{-1}.$$

- Then we get

$$\lambda_{\min}(S) = \left(\sup_{0 \neq q \in W} \inf_{v \in V,\ Bv = q} \frac{(Av, v)}{\|q\|^2} \right)^{-1} = \inf_{0 \neq q \in W} \sup_{v \in V,\ Bv = q} \frac{\|q\|^2}{(Av, v)}.$$

- Moreover, we have

$$\lambda_{\max}(S) = \left(\inf_{0 \neq q \in W} \inf_{v \in V,\ Bv = q} \frac{(Av, v)}{\|q\|^2} \right)^{-1}$$

$$= \left(\inf_{0 \neq v \in V} \frac{(Av, v)}{\|Bv\|^2} \right)^{-1} = \sup_{0 \neq v \in V} \frac{\|Bv\|^2}{(Av, v)}.$$

□

Preconditioned Schur complement

- Let $L\colon W \to W$ be a SPD preconditioner
- We consider the preconditioned Schur complement LS.

Theorem 36 (Spectrum of the preconditioned Schur complement)

The preconditioned Schur complement LS satisfies

$$\lambda_{\min}(LS) = \inf_{0\neq q\in W} \sup_{v\in V,\ Bv=q} \frac{(Lq,q)}{(Av,v)}, \qquad \lambda_{\max}(LS) = \sup_{0\neq v\in V} \frac{(LBv,Bv)}{(Av,v)}.$$

Corollary 37

We set $L = \bar{B}A\bar{B}^t$, where $\bar{B}\colon V \to W$ satisfies $B\bar{B}^t = I$. Then the preconditioned Schur complement LS satisfies

$$\lambda_{\min}(LS) \geq 1, \quad \lambda_{\max}(LS) = \|\bar{B}^t B\|_A^2.$$

[1] J. Mandel and B. Sousedik. BDDC and FETI-DP under minimalist assumptions (Computing 2007).

Table of Contents

- Consider the saddle point problem

$$\begin{bmatrix} A & B^t \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}, \qquad \text{(Saddle)}$$

- **Augmented Lagrangian method**[1]: Given (u^m, p^m), we update (u^{m+1}, p^{m+1}) by

$$\begin{aligned} u^{m+1} &= (A + \epsilon^{-1} B^t B)^{-1} \big(f + \epsilon B^t g - B^t p^m \big), \\ p^{m+1} &= p^m - \epsilon^{-1} \big(g - B u^{m+1} \big), \end{aligned} \qquad m \geq 0.$$

- It is equivalent to the Richardson iteration (step size ϵ^{-1}) on the augmented Schur complement

$$S_\epsilon = B(A + \epsilon^{-1} B^t B)^{-1} B^t.$$

- Hence, the convergence is governed by S_ϵ.

[1] M. Fortin and R. Glowinski. Augmented Lagrangian methods (1983).

- By repeated applications of the auxiliary space lemma, for any $q \in W$ we have

$$(S_\epsilon^{-1} q, q) = \inf_{v \in V,\ Bv=q} ((A + \epsilon^{-1} B^t B)v, v)$$
$$= \inf_{v \in V,\ Bv=q} (Av, v) + \epsilon^{-1}(q, q) = (S^{-1} q, q) + \epsilon^{-1}(q, q).$$

- Thus we obtain the identity[1]

$$S_\epsilon^{-1} = S^{-1} + \epsilon^{-1} I.$$

- The error propagation operator $I - \epsilon^{-1} S_\epsilon$ satisfies

$$\|I - \epsilon^{-1} S_\epsilon\| = 1 - \epsilon^{-1} \lambda_{\min}(S_\epsilon) = \frac{\epsilon}{\epsilon + \lambda_{\min}(S)}.$$

- The extremal eigenvalues of S_ϵ are given by

$$\lambda_{\min}(S_\epsilon) = \frac{\epsilon\, \lambda_{\min}(S)}{\epsilon + \lambda_{\min}(S)}, \qquad \lambda_{\max}(S_\epsilon) = \frac{\epsilon\, \lambda_{\max}(S)}{\epsilon + \lambda_{\max}(S)}.$$

- The augmented Lagrangian method becomes arbitrarily fast as $\epsilon \to 0$.[2]

[1] C.-O. Lee and E.-H. Park. A dual iterative substructuring method with a small penalty parameter (J. Korean Math. Soc. 2017).

[2] Y.-J. Lee, J. Wu, J. Xu, and L. Zikatanov. Robust subspace correction methods for nearly singular systems (Math. Models Methods Appl. Sci. 2007).

Table of Contents

- **Darcy flow**: Linear relationship between the Darcy velocity u and pressure p for flow in porous media

$$\begin{aligned} u + \nabla p &= 0 \quad \text{in } \Omega \\ \operatorname{div} u &= b \quad \text{in } \Omega \\ p &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

- Weak formulation defined on $H(\operatorname{div};\Omega) \times L^2(\Omega)$: find $u \in H(\operatorname{div};\Omega)$ and $p \in L^2(\Omega)$ such that

$$\begin{aligned} \int_\Omega u \cdot v \, dx - \int_\Omega p \operatorname{div} v \, dx &= 0, \\ - \int_\Omega q \operatorname{div} u \, dx &= -\int_\Omega bq \, dx, \end{aligned} \qquad v \in H(\operatorname{div};\Omega), \; q \in L^2(\Omega).$$

- A mixed finite element method is obtained by replacing $H(\operatorname{div};\Omega)$ and $L^2(\Omega)$ with suitable finite element spaces V and W:

$$\begin{bmatrix} M_V & B^t \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} 0 \\ g \end{bmatrix},$$

where the matrices M_V, B, and the vector g are defined by

$$(M_V v, w) = \int_\Omega v \cdot w \, dx,$$
$$(Bv, q) = -\int_\Omega q \operatorname{div} v \, dx, \quad (g, q) = -\int_\Omega bq \, dx.$$

- Continuous vs. discrete divergences

$$B = -M_W \operatorname{div},$$

- Scaling argument (mesh size h)

$$(M_V v, v) \eqsim h^d \|v\|^2, \quad (M_W q, q) \eqsim h^d \|q\|^2.$$

- We assume the discrete Babuška–Brezzi condition holds:

$$\inf_{q \in W} \sup_{v \in V} \frac{(\operatorname{div} v, q)_{L^2}}{\|v\|_{H(\operatorname{div})} \|q\|_{L^2}} \gtrsim 1.$$

- Equivalently, for any $q \in W$, there exists $v \in V$ such that

$$\operatorname{div} v = q, \quad \|q\|_{L^2} \gtrsim \|v\|_{H(\operatorname{div})}.$$

- We analyze the Schur complement $S = BM_V^{-1}B^t$.
- Minimum eigenvalue:

$$\begin{aligned}
\lambda_{\min}(S) &= \inf_{0\neq q\in W} \sup_{v\in V,\ Bv=q} \frac{\|q\|^2}{(M_V v, v)} \\
&= \inf_{0\neq q\in W} \sup_{v\in V, \mathrm{div}\, v=q} \frac{\|M_W q\|^2}{(M_V v, v)} \\
&= h^d \inf_{0\neq q\in W} \sup_{v\in V, \mathrm{div}\, v=q} \frac{(M_W q, q)}{(M_V v, v)} \\
&= h^d \inf_{0\neq q\in W} \sup_{v\in V, \mathrm{div}\, v=q} \frac{\|q\|_{L^2}^2}{\|v\|_{L^2}^2} \\
&\gtrsim h^d.
\end{aligned}$$

- Maximum eigenvalue:

$$
\begin{aligned}
\lambda_{\max}(S) &= \sup_{0 \neq v \in V} \frac{\|Bv\|^2}{(M_V v, v)} \\
&= \sup_{0 \neq v \in V} \frac{\|M_W \operatorname{div} v\|^2}{(M_V v, v)} \\
&\eqsim h^d \sup_{0 \neq v \in V} \frac{\|\operatorname{div} v\|_{L^2}^2}{\|v\|^2} \\
&\lesssim h^{d-2},
\end{aligned}
$$

  where the last inequality follows from the inverse inequality.
- In conclusion, we obtain
$$\kappa(S) \lesssim h^{-2}.$$
- An optimal preconditioner for S can be constructed by exploiting its spectral equivalence with a certain discretization of the Poisson problem.[1]

[1] T. Rusten, P. Vassilevski, and R. Winther. Interior penalty preconditioners for mixed finite element approximations of elliptic problems (Math. Comp. 1996).

- **Stokes equations**: Incompressible Stokes equations with the homogeneous Dirichlet boundary condition

$$\begin{aligned} -\Delta u + \nabla p &= f && \text{in } \Omega, \\ \operatorname{div} u &= 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

- Weak formulation defined on $H_0^1(\Omega)^d$ and $L_0^2(\Omega)$: find $(u, p) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$ such that

$$\begin{aligned} \int_\Omega \nabla u \cdot \nabla v \, dx - \int_\Omega p \operatorname{div} v \, dx &= \int_\Omega f \cdot v \, dx, \\ -\int_\Omega q \operatorname{div} u \, dx &= 0, \end{aligned} \qquad v \in H_0^1(\Omega)^d, \ q \in L_0^2(\Omega).$$

- A mixed finite element method is obtained by replacing $H_0^1(\Omega)^d$ and $L_0^2(\Omega)$ with suitable finite element spaces V and W.
- We obtain (Saddle) with

$$(Av, w) = \int_\Omega \nabla v \cdot \nabla w \, dx,$$

$$(Bv, q) = -\int_\Omega q \operatorname{div} v \, dx, \quad (f, v) = \int_\Omega f \cdot v \, dx.$$

- Continuous vs. discrete divergences

$$B = -M_W \operatorname{div},$$

- Scaling argument (mesh size h)

$$(M_V v, v) \eqsim h^d \|v\|^2, \quad (M_W q, q) \eqsim h^d \|q\|^2.$$

- We assume the discrete Babuška–Brezzi condition holds:

$$\inf_{q \in W} \sup_{v \in V} \frac{(\operatorname{div} v, q)_{L^2}}{\|v\|_{H^1} \|q\|_{L^2}} \gtrsim 1,$$

- Equivalently, for any $q \in W$, there exists $v \in V$ such that

$$\operatorname{div} v = q, \quad \|q\|_{L^2} \gtrsim \|v\|_{H^1}.$$

- We analyze the Schur complement $S = BA^{-1}B^t$.
- Minimum eigenvalue:

$$\begin{aligned}
\lambda_{\min}(S) &= \inf_{0\neq q\in W} \sup_{v\in V,\ Bv=q} \frac{\|q\|^2}{(Av,v)} \\
&= \inf_{0\neq q\in W} \sup_{v\in V,\ \operatorname{div} v=q} \frac{\|M_W q\|^2}{(Av,v)} \\
&\eqsim h^d \inf_{0\neq q\in W} \sup_{v\in V,\ \operatorname{div} v=q} \frac{(M_W q, q)}{(Av,v)} \\
&= h^d \inf_{0\neq q\in W} \sup_{v\in V,\ \operatorname{div} v=q} \frac{|q|_{H^1}^2}{\|v\|_{L^2}^2} \\
&\gtrsim h^d.
\end{aligned}$$

- Maximum eigenvalue:

$$\begin{aligned} \lambda_{\max}(S) &= \sup_{0 \neq v \in V} \frac{\|Bv\|^2}{(Av, v)} \\ &= \sup_{0 \neq v \in V} \frac{\|M_W \operatorname{div} v\|^2}{(Av, v)} \\ &\approx h^d \sup_{0 \neq v \in V} \frac{(M_W \operatorname{div} v, \operatorname{div} v)}{(Av, v)} \\ &= h^d \sup_{0 \neq v \in V} \frac{\|\operatorname{div} v\|_{L^2}^2}{|v|_{H^1}^2} \\ &\lesssim h^d. \end{aligned}$$

- In conclusion, we obtain

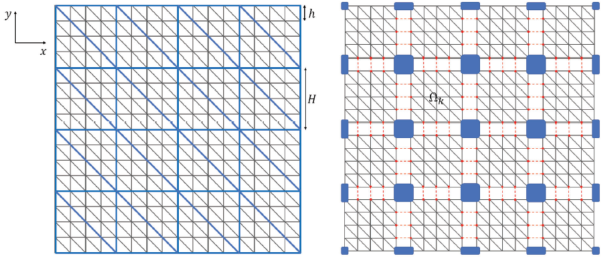
$$\kappa(S) \lesssim 1.$$

- The conjugate gradient method for solving the dual problem converges uniformly with respect to h.[1]

[1] R. Verfürth. A combined conjugate gradient–multi-grid algorithm for the numerical solution of the Stokes problem (IMA J. Numer. Anal. 1984).

Table of Contents

- **FETI-DP**[1,2]: One of the most broadly used nonoverlapping domain decomposition methods (FETI, FETI-DP, BDD, BDDC, . . .)
- A global problem is partitioned into smaller subproblems posed on subdomains, and continuity across subdomain interfaces is enforced by means of Lagrange multipliers.



(A) Triangulations $\mathcal{T}_h$ and $\mathcal{T}_H$ (B) Dual-primal domain decomposition [3]

[1] C. Farhat, M. Lesoinne, P. LeTallec, K. Pierson, and D. Rixen. FETI-DP: a dual–primal unified FETI method—part I: A faster alternative to the two-level FETI method (Internat. J. Numer. Methods Engrg. 2001).

[2] A. Klawonn, O.B. Widlund, and M. Dryja. Dual–primal FETI methods for three-dimensional elliptic problems with heterogeneous coefficients (SIAM J. Numer. Anal. 2002).

[3] C.-O. Lee and **JP**. A dual–primal finite element tearing and interconnecting method for nonlinear variational inequalities utilizing linear local problems (Internat. J. Numer. Methods Engrg. 2021).

FETI-DP: Problem setting

- Model problem: Poisson equation defined on a bounded polygonal domain $\Omega \subset \mathbb{R}^2$:

$$
\begin{aligned}
-\Delta u &= f \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial\Omega.
\end{aligned}
$$

- Nonoverlapping domain decomposition: The domain Ω is decomposed into J nonoverlapping polygonal subdomains $\{\Omega_j\}_{j=1}^J$ with characteristic subdomain diameter $H > 0$.

- Subdomain interfaces: let $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$ denote the interface between adjacent subdomains, and define $\Gamma = \bigcup_{i<j} \Gamma_{ij}$.

- Local triangulations: $\mathcal{T}_h^i$ and $\mathcal{T}_h^j$ share nodal points along the interface Γ_{ij}.

- Local boundary FE spaces: On each Ω_j, we consider the space of continuous, piecewise linear finite elements on $\mathcal{T}_h^j$ that vanish on $\partial\Omega_j \cap \partial\Omega$, and denote its restriction to the interface $\partial\Omega_j$ by V_j.

- Product space: $V = \prod_{j=1}^J V_j$.

- Note that functions in V are, in general, discontinuous across Γ.

FETI-DP: Problem setting

FETI[1] constrained optimization problem

$$\min_{v=(v_j)_{j=1}^J \in V} \left\{ \frac{1}{2}(Sv, v) - (f, v) \right\} \quad \text{subject to} \quad Bv = 0$$

- S and f are defined by

$$(Sv, w) = \sum_{j=1}^{J} \int_{\Omega_j} \nabla \mathcal{H}_j v_j \cdot \nabla \mathcal{H}_j w_j \, dx, \quad (f, v) = \sum_{j=1}^{J} \int_{\Omega_j} f \, \mathcal{H}_j v_j \, dx.$$

- $\mathcal{H}_j$ denotes the discrete harmonic extension in Ω_j associated with $\mathcal{T}_h^j$.
- B is a full-rank matrix with entries 0 and ± 1 enforcing continuity along Γ.

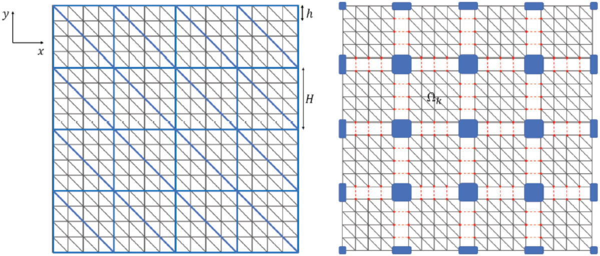
FETI saddle point formulation

$$\begin{bmatrix} S & B^t \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ \lambda \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}.$$

- We use the Lagrange multiplier $\lambda \in W$ to deal with the constraint.
- S is semi-SPD, owing to subdomains Ω_j that do not intersect $\partial\Omega$.

[1] C. Farhat and F.-X. Roux. A method of finite element tearing and interconnecting and its parallel solution algorithm (Internat. J. Numer. Methods Engrg. 1991).

- While the FETI formulation enforces continuity along the entire interface Γ through the constraint, FETI-DP instead imposes continuity at subdomain corners directly by restricting the solution space to a subspace $\tilde{V} \subset V$.
- The space $\tilde{V}$ consists of functions in V that are continuous at subdomain corners.
- Continuity along the interior of each subdomain edge is enforced by constraints.



(A) Triangulations $\mathcal{T}_h$ and $\mathcal{T}_H$ (B) Dual-primal domain decomposition 3

FETI-DP: Problem setting

FETI-DP[1] constrained optimization problem

$$\min_{v=(v_j)_{j=1}^J \in \tilde{V}} \left\{ \frac{1}{2}(\tilde{S}v, v) - (f, v) \right\}, \quad \text{subject to} \quad Bv = 0,$$

- $\tilde{S}$ and $\tilde{f}$ are defined analogously.
- B is a full-rank matrix with entries 0 and ± 1 that enforces continuity along the interior of subdomain edges.

FETI-DP saddle point formulation

$$\begin{bmatrix} \tilde{S} & B^t \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ \lambda \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}.$$

- We use the Lagrange multiplier $\lambda \in W$ t deal with the constraint.
- Different from the FETI formulation $\tilde{S}$ is SPD.

FETI-DP dual problem

$$F\lambda = d, \quad \text{where} \quad F = B\tilde{S}^{-1}B^t, \quad d = \tilde{S}^{-1}f.$$

[1] C. Farhat, M. Lesoinne, P. LeTallec, K. Pierson, and D. Rixen. FETI-DP: a dual–primal unified FETI method—part I: A faster alternative to the two-level FETI method (Internat. J. Numer. Methods Engrg. 2001).

Continuity matrix B

- The matrix B satisfies the following identity[1]:

$$BB^t = 2I.$$

- If we define $\bar{B} = \frac{1}{2}B$, then $\bar{B}^t$ is a right inverse of B.

Poincaré-type inequalities
As FETI-DP involves functions that are continuous at subdomain corners, we require certain Poincaré-type inequalities associated with subdomain corners.[1,2]

Lemma 38

For each subdomain $\Omega_j \subset \mathbb{R}^2$, the following estimates hold:

$$\left.\begin{aligned} \|v_j - I_H v_j\|_{L^2(\partial\Omega_j)}^2 &\lesssim H\left(1 + \log\frac{H}{h}\right)|v_j|_{H^{1/2}(\partial\Omega_j)}^2, \\ \sum_{e:\text{ edge of }\Omega_j} |I_e^0(v_j - I_H v_j)|_{H^{1/2}(\partial\Omega_j)}^2 &\lesssim \left(1 + \log\frac{H}{h}\right)^2 |v_j|_{H^{1/2}(\partial\Omega_j)}^2, \end{aligned}\right\} \quad v_j \in V_j,$$

where I_e^0 is the extension-by-zero operator.

[1] J. Mandel and R. Tezaur. On the convergence of a dual–primal substructuring method (Numer. Math. 2001).

[2] C.-O. Lee, E.-H. Park, and **JP**. Corrigendum to "A dual iterative substructuring method656 with a small penalty parameter" (J. Korean Math. Soc. 2021).

- We analyze the FETI-DP dual operator $F = B\tilde{S}^{-1}B^t$.

- Minimum eigenvalue:

$$\begin{aligned}
\lambda_{\min}(F) &= \inf_{0\neq\lambda\in W} \sup_{v\in\tilde{V},\ Bv=\lambda} \frac{\|\lambda\|^2}{(\tilde{S}v, v)} \\
&\geq \inf_{0\neq\lambda\in W} \frac{\|\lambda\|^2}{(\tilde{S}\bar{B}^t\lambda, \bar{B}^t\lambda)} \\
&\gtrsim \lambda_{\max}(\tilde{S})^{-1} \\
&\gtrsim 1.
\end{aligned}$$

- Maximum eigenvalue: since $\mathcal{R}(I_H) \subset \mathcal{N}(B)$, for any $v = (v_j)_{j=1}^J \in \tilde{V}$,

$$\begin{aligned} \|Bv\|^2 &= \|B(v - I_H v)\|^2 \\ &\lesssim \|v - I_H v\|^2 \\ &\approx h^{-1} \sum_{j=1}^{J} \|v_j - I_H v_j\|_{L^2(\partial\Omega_j)}^2 \\ &\lesssim \frac{H}{h}\left(1 + \log\frac{H}{h}\right) \sum_{j=1}^{J} |v_j|_{H^{1/2}(\partial\Omega_j)}^2 \\ &= \frac{H}{h}\left(1 + \log\frac{H}{h}\right)(Sv, v), \end{aligned}$$

- Hence, we deduce

$$\lambda_{\max}(F) \lesssim \frac{H}{h}\left(1 + \log\frac{H}{h}\right).$$

- In conclusion, we obtain[1]

$$\kappa(F) \lesssim \frac{H}{h}\left(1 + \log\frac{H}{h}\right).$$

[1] C.-O. Lee and E.-H. Park. A dual iterative substructuring method with a small penalty parameter (J. Korean Math. Soc. 2017).

- Next, we consider the following Dirichlet preconditioner:

$$L_{\mathrm{DP}} = \bar{B}\tilde{S}\bar{B}^t,$$

- Minimum eigenvalue: By Corollary 37, we have

$$\lambda_{\min}(L_{\mathrm{DP}}F) \geq 1.$$

- Maximum eigenvalue: for any $v = (v_j)_{j=1}^J \in \tilde{V}$,

$$\begin{aligned}(\tilde{S}\bar{B}^t Bv, \bar{B}^t Bv) &= (\tilde{S}\bar{B}^t B(v - I_H v), \bar{B}^t B(v - I_H v)) \\ &= \sum_{j=1}^J |\bar{B}^t B(v - I_H v)|^2_{H^{\frac{1}{2}}(\partial\Omega_j)} \\ &\lesssim \sum_{j=1}^J \sum_{e:\ \text{edge of } \Omega_j} \left| I_e^0 (v_j - I_H v_j) \right|^2_{H^{\frac{1}{2}}(\partial\Omega_j)} \\ &\lesssim \left(1 + \log \frac{H}{h}\right)^2 \sum_{j=1}^J |v_j|^2_{H^{1/2}(\partial\Omega_j)} \\ &= \left(1 + \log \frac{H}{h}\right)^2 (\tilde{S}v, v),\end{aligned}$$

- Consequently, we have

$$\lambda_{\mathsf{max}}(L_{\mathrm{DP}}F) \lesssim \left(1 + \log \frac{H}{h}\right)^2 .$$

- Finally, we obtain the following condition number bound for the preconditioned operator $L_{\mathrm{DP}}F$[1]:

$$\kappa(L_{\mathrm{DP}}F) \lesssim \left(1 + \log \frac{H}{h}\right)^2 .$$

Remark 3 (Other nonoverlapping domain decomposition methods)

Closely related nonoverlapping domain decomposition methods, such as FETI, BDD, and BDDC[2][3][4], can be analyzed within our framework by essentially the same arguments used for FETI-DP.

[1] J. Mandel and R. Tezaur, On the convergence of a dual–primal substructuring method (Numer. Math. 2001).

[2] J. Mandel, C.R. Dohrmann, and R. Tezaur. An algebraic theory for primal and dual substructuring methods by constraints (Appl. Numer. Math. 2005).

[3] J. Li and O.B. Widlund. FETI-DP, BDDC, and block Cholesky methods (Internat. J. Numer. Methods Engrg. 2006).

[4] S.C. Brenner and L.-Y. Sung, BDDC and FETI-DP without matrices or vectors (Comput. Methods Appl. Mech. Engrg. 2007).

- **Saddle point problems and Schur complements**
  - Saddle point problem

$$\begin{bmatrix} A & B^t \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}, \qquad \text{(Saddle)}$$

  - Schur complement system

$$Sp = d, \quad \text{where } S = BA^{-1}B^t \colon W \to W, \; d = BA^{-1}f - g. \qquad \text{(Schur)}$$

    has an auxiliary space structure.
  - Schur complements are central in design and analysis of iterative methods.
- **Sharp estimates for Schur complements**
  - Sharp spectral estimates using the auxiliary space theory

$$\lambda_{\min}(S) = \inf_{0 \neq q \in W} \sup_{v \in V, \; Bv = q} \frac{\|q\|^2}{(Av, v)}, \quad \lambda_{\max}(S) = \sup_{0 \neq v \in V} \frac{\|Bv\|^2}{(Av, v)}.$$

- **Applications**
  - Augmented Lagrangian method
  - Mixed finite element methods
  - Nonoverlapping domain decomposition methods

Thank you for your attention!

References

1. **JP**. Unified analysis of saddle point problems via auxiliary space theory (2025+).
2. **JP** and Jinchao Xu. Auxiliary space theory for the analysis of iterative methods for semidefinite linear systems (2025+).
3. Jinchao Xu and Ludmil Zikatanov. Algebraic multigrid methods (Acta Numer. 2017).